3 *One-Dimensional Signal Processing*

> Today, "flow", or its equivalent, "continuity", is so unclear as to be almost devoid of meaning.
>
> E.T. Bell, *Men of Mathematics*

3.1 Introduction

Although continuous domain filter theory has an earlier heritage, many engineering problems necessarily deal with discrete data due to the ubiquitous use of digital computers. The underlying theory of random processes and Fourier transforms allows a great deal of information to be extracted from time series data. Transforming to the frequency or spectral domains allows processes obscured in the temporal domain to be easily identified. The Fast Fourier Transform (FFT) is a standard tool in many disciplines, but to solve problems dealing with random processes, a more in-depth knowledge of the Fourier transform and its statistical properties is required. Furthermore, different structures of the Fourier transform yield additional insight into the underlying operations and properties of the transform. The present chapter covers important properties of Fourier theory, but the content is limited to ideas either necessary to understand or to generalize a later problem.

3.2 Fourier Series

A periodic function f(t) can be written as a linear superposition of complex exponentials

$$f(t) = \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_0 t} \qquad (3.1)$$

where

$$\omega_0 = \frac{2\pi}{T} \qquad (3.2)$$

and the weighting coefficients are

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt \tag{3.3}$$

The Fourier series is a series of equidistant impulses in the spectral domain, each weighted by its respective coefficient C_n (Papoulis, 1962). The expansion rests on the orthogonality of the trigonometric sine and cosine functions. If the expansion is truncated before all possible harmonic components have been estimated, the truncated expansion corresponds to a Gaussian least squares fit of the data up to that harmonic.

The Fourier series suffers from several characteristics that limit its application to many engineering problems encountered in practice. First, the function must be periodic. The Discrete Fourier Transform (DFT) duplicates the Fourier series concept for aperiodic functions that in reality have no true Fourier series representation, exploiting the periodic replication of a finite time series. Second, the Fourier series can only be used as a theoretical guide when analyzing natural phenomena stemming from random processes.

3.3 <u>Fourier Transform</u>

The Fourier transform is usually derived as a special case of the Fourier series, but the derivation can also proceed from the Fourier transform to the Fourier series. Random processes necessitate looking at the Fourier transform as the more general solution, since the Fourier series really represents an ideal rarely found in practice. In this section the Fourier transform is derived and the Fourier series is placed in perspective as the limit of a sequence of transforms.

Physical possibility is a valid sufficient condition for the existence of a Fourier transform (Bracewell, 1986). Any reasonable, arbitrary signal can be represented as the superposition of weighted complex exponentials. The Fourier transform extends the applicability of Fourier theory by allowing any function to be transformed into its spectral components. Links exist between operations performed in one domain and the resulting operations in the other domain. Sampling and periodic replication are an important link between the two domains. Sampling in time implies periodic replication in frequency, and periodic replication in time implies sampling in frequency.

The Fourier transform will be introduced first in the continuous domain, and the Fourier integral discussed. The Continuous Time Fourier Transform (CTFT), although not applicable when dealing with discrete data, will be useful when discussing smoothing kernels and the underlying properties of Fourier domain operations. The Discrete Time Fourier Transform (DTFT) will then be discussed for its additional usefulness when dealing with time limited signals. Last, the Discrete Fourier Transform (DFT) will be introduced. The Fast Fourier Transform (FFT) will be covered briefly, with no details pertaining to the efficient implementation alternatives.

3.3.1 The Continuous Time Fourier Transform (CTFT)

The CTFT has several essentially equivalent definitions, with one definition given by

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt \tag{3.4}$$

where $F(\omega)$ is the Fourier Transform of $f(t)$. The inverse CTFT allows the synthesis of $f(t)$ from its spectral components as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{j\omega t}d\omega \tag{3.5}$$

The symmetry of the two domains is evident, and the factor of $1/(2\pi)$ can be introduced in several ways.

The CTFT allows continuous time measurements to be transformed into a continuous frequency spectrum. The infinite limits on the integrals assume infinite measurements from the past and future are available, which is never possible in practice. The infinite dimensional frequency space also introduces obvious questions regarding the computation of the spectrum. In geotechnical engineering problems, the CTFT is rarely implemented, since processes are usually sampled discretely. The CTFT offers the clearest view into the nature of the transform, and allows much easier interpretation of the theory and spectral properties of the underlying transform.

3.3.2 The Discrete Time Fourier Transform (DTFT)

The DTFT is the primary theoretical guide for investigating Fourier transform spectral operators, and, in practice, the DTFT is also very useful when dealing with time-limited processes. Sampling in time does not necessarily prohibit the engineer from obtaining a continuous spectrum. The DTFT is the form of the Fourier transform that allows discrete time data to be transformed into a continuous spectrum, and is given by

$$F(\omega) = \sum_{n=-\infty}^{+\infty} f[n]e^{-j\omega n} \tag{3.6}$$

where $f[n]$ equals the samples of the continuous time process $f(t)$ at sampling indices n. Notice the need for an infinite amount of data in the summation, which is impossible unless the signal is time-limited. When the process is time-limited, as in a transient or impulse response, the process $f[n]$ is measured in its entirety, since it is zero before and after the transient. The DTFT spectrum is periodic with a period of 2π. The actual ranges of the spectrum are determined by the sampling rate f_S.

The synthesis formula for the DTFT is given by

$$f[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} F(\omega) e^{j\omega n} d\omega \tag{3.7}$$

Notice that only one period of the spectrum needs to be integrated due to the periodicity of the DTFT frequency spectrum.

3.3.3 The Discrete Fourier Transform (DFT)

The DFT is the form of the Fourier transform most often implemented in applied engineering, due to its simplicity and efficiency. Entire books have been written on the subject, and a larger treatise should be consulted for further information regarding the properties and theory of the DFT, for example, Oppenheim and Schafer (1989).

Conceptually, the DFT samples the DTFT at N equally spaced points along the unit circle in the z-transform complex plane. The DFT is most often written as

$$F[k] = \sum_{n=0}^{N-1} f[n] W_N^{kn} \tag{3.8}$$

where N equals the total number of samples collected, k equals the spectral domain index, and

$$W_N = e^{-j(2\pi / N)} \tag{3.9}$$

The synthesis equation is

$$f[n] = \frac{1}{N} \sum_{k=0}^{N-1} F[k] W_N^{-kn} \tag{3.10}$$

The DFT replicates the temporal sequence periodically to obtain the spectral components. Therefore, since the temporal signal is discrete and periodic, the frequency spectrum is discrete and periodic, with a period N.

The term Fast Fourier Transform (FFT) encompasses many efficient algorithms available for computing the entire DFT of a sequence. Through exploitation of the periodicity and symmetry of the sequence W_N, the class of FFT algorithms can be orders of magnitudes more numerically efficient in terms of multiplications and additions compared to competing algorithms. The FFT decomposes the DFT of a sequence of length N into successively smaller and smaller DFTs. The most common FFT algorithms fall under the categories of decimation-in-time and decimation-in-frequency. For a more complete discussion of the FFT, see Oppenheim and Schafer (1989). Although the FFT allows considerable computational savings when calculating all N points of the DFT, other algorithms are more efficient at calculating only a portion of the spectrum, such as the Goertzel and chirp transform algorithms (Oppenheim and Schafer, 1989).

3.3.4 Generalized Functions and Transforms in the Limit

Many mathematical properties are defined, and only exist, in terms of limits. The following discussion introduces the concept of a limit of a sequence of functions and generalized functions, which will aid considerably in the interpretation and understanding of spectral operators and Fourier transforms. The discussion is brief, simplified, and intended as an introduction for the geotechnical engineering field, in which generalized functions are not commonly employed. For more in-depth discussions, especially relating to engineering analysis, see Bracewell (1986) and Papoulis (1962). After introducing the concept of generalized functions, the Fourier series is placed in perspective as the limit of a sequence of transforms.

3.3.4.1 Generalized Functions

The Dirac delta impulse is an important theoretical concept in many areas of engineering, including point sources and point masses, and is defined as (Bracewell, 1986)

$$\left.\begin{array}{ll} \delta(x) = 0 & x \neq 0 \\ \displaystyle\int_{-\infty}^{+\infty} \delta(x)\,dx = 1 & \end{array}\right\} \qquad (3.11)$$

The Dirac delta falls outside the realm of analysis typically encompassed by the term function, and instead falls into the category of generalized functions or the theory of distributions (Papoulis, 1962). The Dirac delta will first be interpreted as the limit of a series of functions, using the rectangular pulse, and the discussion will be extended to the more encompassing meaning of a generalized function.

The simplest interpretation of the Dirac delta impulse is as the limit of a series of increasingly brief, increasingly strong rectangular pulses, as shown in Figure 3.1. As defined by the integral in Equation 3.11, each impulse shown in Figure 3.1 has an area equal to one. The rectangular pulse is not the only function which produces a sequence tending to the Dirac delta impulse in the limit. Gaussian, triangular, and sinc functions all yield a Dirac delta impulse in the limit, but the properties of the different functions, such as existence of derivatives, make them useful in different types of analyses (Bracewell, 1986).

Bracewell (1986) defines a generalized function as a regular sequence of particularly well-behaved functions. The definition imposes restrictive conditions on the derivatives and asymptotic behavior of the members of the sequence of functions. In the strict definition, derivatives of all orders at all points of the function must exist and an asymptotic decay condition must be satisfied. Several different sequences of regular, particularly well-behaved functions may converge to the same generalized function, and the generalized function actually consists of the class of all convergent sequences of functions. Regarding the most common functions of interest to signal processing, the rectangular pulse fails the strict definition of a generalized function due to failure to meet the derivative existence requirements, while the sinc function fails to meet the asymptotic decay criterion. In practice, the use of the sinc function as a generalized function is acceptable due to the finite length of commonly encountered signals (Bracewell, 1986). The sinc function is the most commonly used function in signal processing whose properties tend to the Dirac delta in the limit.

3.3.4.2 Fourier Series as the Limit of a Sequence of Fourier Transforms

A sequence must be absolutely summable to yield a uniformly convergent Fourier transform (Oppenheim and Schafer, 1989), and therefore, a periodic function does not have a Fourier transform. Nevertheless, periodic functions are considered to have line spectrums, and Fourier transform theory is broadened to handle functions with a line spectrum by introducing the concept of transforms in the limit (Bracewell, 1986).

A transform in the limit is attained by modifying the periodic function with a factor that may yield a function that possesses a Fourier transform (Bracewell, 1986). The following discussion takes a different approach to emphasize a few of the practical considerations concerning Dirac deltas, windows, and Fourier transforms, and makes no claim of mathematical rigor. Figure 3.2 shows the sequence of Fourier transforms of a single, 10 Hz sinusoid as the window length increases from 1 second to 80 seconds. As the

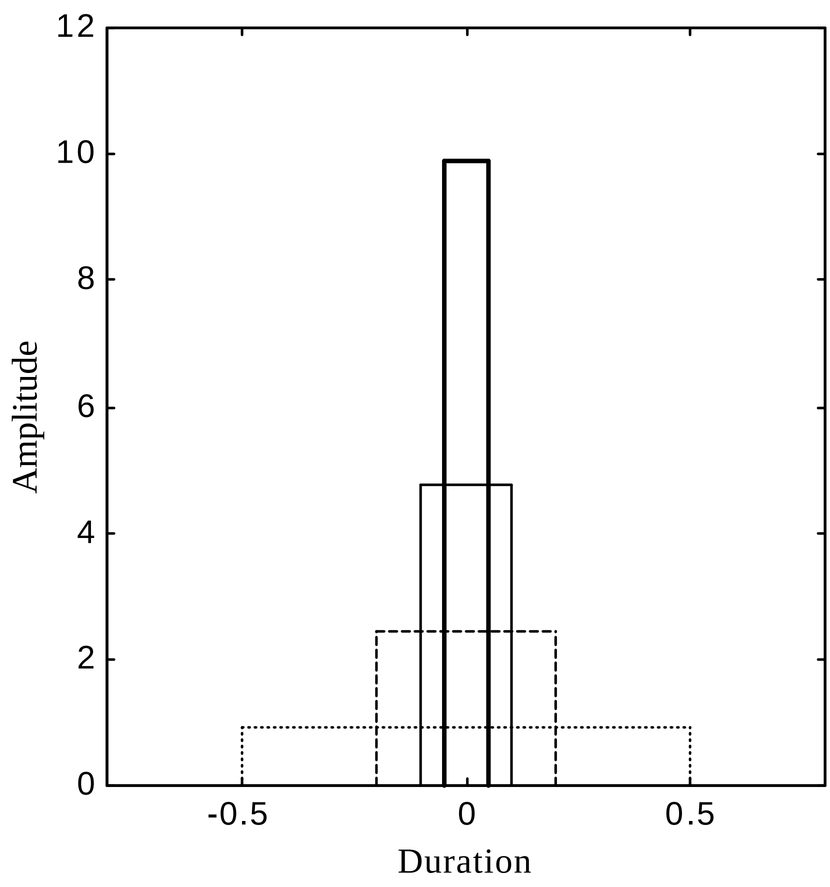

Figure 3. 1 Dirac Delta Impulse as the Limit of a Sequence of Rectangular Pulses. A sequence of increasingly brief rectangular impulses, all with area equal to one, tending to the infinitely brief, infinitely high Dirac delta impulse.

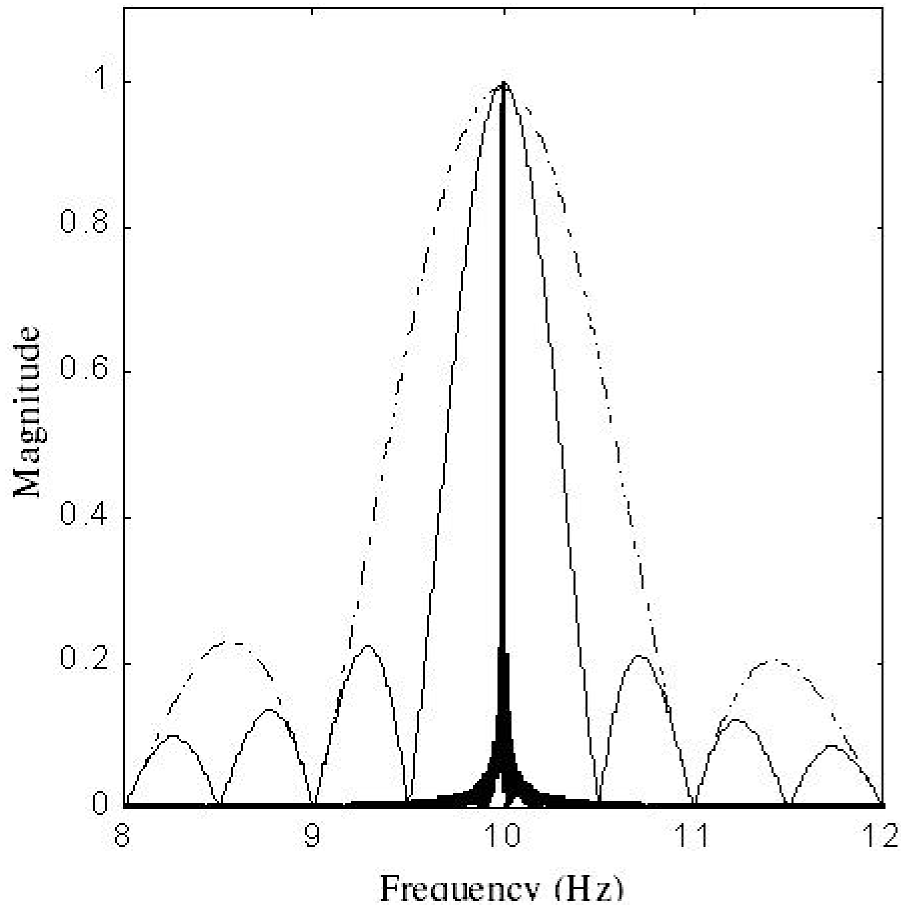

Figure 3. 2 Sequence of Fourier Transforms of a Single Sinusoid. The sequence of Fourier transforms for a single, 10 Hz sinusoid are shown for window lengths of 1 second (dashed line), 2 seconds (light solid line), and 80 seconds (heavy solid line).

window length increases, the spectrum estimate progressively becomes narrower and narrower, and if an infinite amount of data were available, the estimate would approach a line spectrum in the limit. Additionally, regardless of the window type, as the amount of data tends toward infinity, the smoothing kernel will approach a delta impulse.

3.3.5 Summary of the Fourier Transform

Table 3.1 summarizes the forms of the Fourier transform covered in Sections 3.3.1 to 3.3.3. Although the FFT is used the most often in practice due to numerical efficiency, the DTFT and the CTFT provide the most convenient framework for discussions about the spectral properties of operators.

Table 3.1 Fourier Series and Forms of the Fourier Transform

	Equations	Temporal & Spectral Domains	Comments
Fourier Series	$f(t) = \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_0 t}$ $C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt$	*Spectral* Line spectrum	
CTFT	$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$ $f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{j\omega t}d\omega$	*Temporal* Continuous *Spectral* Continuous	
DTFT	$F(\omega) = \sum_{n=-\infty}^{+\infty} f[n]e^{-j\omega n}$ $f[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} F(\omega)e^{j\omega n}d\omega$	*Temporal* Discrete *Spectral* Continuous Periodic	• Useful for analysis of spectral operator properties • Useful for transient signal analysis
DFT	$F[k] = \sum_{n=0}^{N-1} f[n]W_N^{kn}$ $f[n] = \frac{1}{N}\sum_{k=0}^{N-1} F[k]W_N^{-kn}$	*Temporal* Discrete Periodic *Spectral* Discrete Periodic	• Samples DTFT along unit circle • Efficient implementation through FFT

3.4 Temporal Domain

The most common one-dimensional signal processing problem involves temporal data. The format of the current section will highlight the most salient features of time domain measurements, while also setting a foundation for a parallel interpretation in the multidimensional case. The sampling characteristics control the temporal lag domain, which limits the frequency range available for spectrum estimation. The weighting function offers the greatest control over the spectrum estimation problem, as shown in Section 3.5. The primary function of interest in time domain measurements is the autocorrelation function, which describes signal correlation over different temporal lags.

3.4.1 Temporal Lag Domain

The procedure used to experimentally measure a temporal random process exerts a great influence on the later power spectrum estimation problem. The highest frequency available for power estimation is determined by the sampling frequency, and the total number of samples N determines the ability to resolve different frequencies. The lag domain refers to the sampling characteristics devoid of any actual experimental information. The sampling rate t_S controls the redundancy of various temporal lags, and the total length of data $(N-1)*t_S$ controls the longest temporal lag available. An inherent tradeoff exists between N and t_S, which is described mathematically by the uncertainty relation. In typical temporal measurements, the sampling rate t_S is constant. Therefore, the temporal lag domain forms a triangle centered at a temporal lag equal to zero. Figure 3.3 displays an example of the temporal lag domain for t_S equal to 0.1 seconds and N = 10.

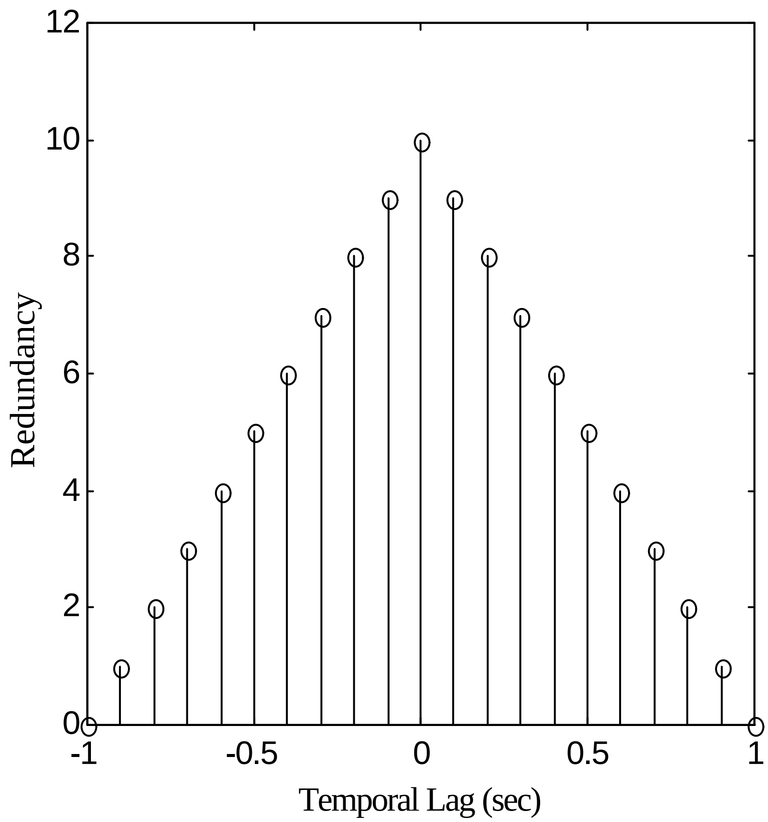

Figure 3. 3 Temporal Lag Domain for t_S = 0.1 Seconds and N = 10

3.4.2 Uncertainty Relation

The bandwidth-duration product of a signal cannot be less than a certain minimum value, preventing arbitrary specification of signals on the time-frequency plane. The signals may be brief and wideband (e.g. a temporal impulse and a white noise spectrum) or monochromatic and temporally persistent (e.g. a temporally periodic function and a line spectrum). The usual form of the uncertainty relation is

$$\Delta t \Delta f \geq \frac{1}{4\pi} \tag{3.12}$$

where Δt and Δf represent the equivalent duration and bandwidth (Bracewell, 1986). Therefore, an inherent tradeoff exists between the temporal and frequency domains.

3.4.3 Nyquist Sampling Rate and Aliasing

If sampled at a high enough rate, any reasonable, band-limited temporal signal can be completely characterized in the frequency domain. The necessary sampling rate, called the Nyquist rate, equals two times the Nyquist frequency. The Nyquist frequency of a band-limited signal $F(\omega)$ is given by

$$F(\omega) = 0 \qquad \text{for } |\omega| > \omega_{Nyquist} \tag{3.13}$$

If the process is sampled below the Nyquist rate, higher frequency components alias into lower frequencies.

In reality, the frequency content of experimental measurements is not known until after data collection. Implementation of analogue anti-aliasing filters ameliorates aliasing in the temporal data. The concept of aliasing in the one-dimensional case is introduced in anticipation of the much more difficult question of spatial aliasing, considered in the next chapter.

3.4.4 Weighting Function

A weighting vector w[n], or window function, applied in the temporal domain affects the properties of the spectral operators. The spectral domain effects are discussed in Section 3.5. The effects of different weighting vectors may be analyzed without consideration of experimental measurements. Some common weight vectors applied in the temporal domain are the rectangular, Bartlett, Hanning, and Hamming windows. The temporal weighting vector changes the relative importance of long versus short temporal lags. The tapering at larger lags decreases the relative weight given to measurements at long lags, which intuitively seems correct, since fewer samples are obtained at longer lags versus shorter lags.

3.4.5 Autocorrelation Function

The primary function of interest to characterize a temporal sequence is the autocorrelation function. The autocorrelation describes the correlation of the measured process with itself at various temporal lags. For a sampling rate t_s, the autocorrelation function covers lags up to the total time length of the data (N-1)·t_s, where N is the total number of samples. The shortest and longest lags contained in the autocorrelation function control the ability to resolve different frequencies and the maximum frequency at which the power spectral density can be estimated.

The autocorrelation estimated from real-valued measurements equals

$$r_s(\ell) = \frac{1}{N}\sum_{n=0}^{N-1} s(n+\ell)s(n) \qquad \ell = 0 \text{ to } N-1 \tag{3.14}$$

where s(n) = signal measured at time index n, and $r_s(\ell)$ = the estimated autocorrelation function at time lag index ℓ. $r_s(\ell)$ for negative lags ℓ is determined by symmetry (Hayes, 1996).

3.5 Spectral Domain

The ideal spectrum estimating filter would be a Dirac delta impulse at the frequency of interest, which would perfectly sift out a single frequency. The sifting property (Bracewell, 1986) of the delta function is not obtainable in practice, so engineers must accept filters which approach the characteristics of the delta generalized function in the limit. The compromise is usually a form of the sinc function, representing a bandpass filter with its center frequency matched to the frequency of interest

Estimating the spectrum of a temporal process may be viewed as designing a bank of bandpass filters, with each filter centered at the frequency of interest. Mainlobe width controls resolution between closely spaced frequencies, and sidelobe height controls energy leakage from more remote parts of the spectrum. The temporal sampling and weighting characteristics control the properties of the spectral filter, and when viewed as a filter design problem, the temporal weighting function is the primary design variable.

The spectral filter design problem requires designing an optimum filter for extraction of specific information relevant to a specific problem. The optimum design will vary, depending on the underlying process, e.g. bandwidth, frequency component separation, and noise power, and the goals of the design. Overall, a filter may be characterized as good if it has a narrow mainlobe and good control of sidelobe heights. The following sections will discuss the ideal spectral filter, which requires infinite information and a deterministic process. Since infinite information is never available, the optimum design problem will be generalized through the temporal weighting vector.

3.5.1 Ideal Smoothing Kernel

Assume perfect, infinite-length measurements of a single, unit amplitude, single frequency ω_0 temporal wave are made with no noise. The transform of the process would produce a line spectrum. The Fourier transform equals

$$F(\omega) = \int_{-\infty}^{+\infty} \delta(\omega - \omega_0)\, d\omega \tag{3.15}$$

i.e. a unit impulse at the frequency ω_0. The Dirac delta δ perfectly sifts out the frequency of interest and represents the ideal smoothing kernel. Graphically, the results are shown in Figure 3.4.

3.5.2 Integral Transform Smoothing Kernel

In reality, infinite length, perfect experimental measurements and the ideal sifting property of the Dirac delta are not obtainable. Engineers must compromise and accept a filter approaching the sifting property of the Dirac delta in the limit. The general Fourier transform (Equation 3.4) maps time to frequency with the following integral equation

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt \qquad (3.16)$$

In practice, the underlying random process and the sampling rate t_S constrain the available data length. The resulting discrete transform is given by

$$\hat{F}(\omega) = \sum_{n=0}^{N-1} f[n]e^{-j\omega n} \qquad (3.17)$$

where the hat indicates estimation, and N = total number of samples. The equation can be rewritten as

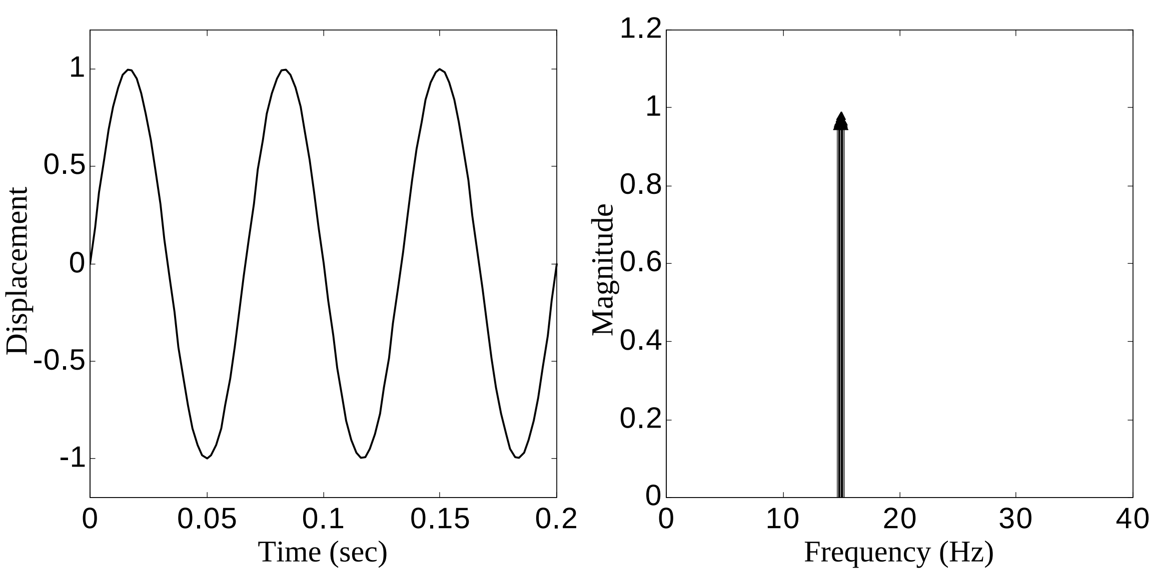

Figure 3. 4 Graphical Representation of a Line Spectrum. An infinite duration signal (left panel) and the corresponding line spectrum (right panel).

$$\hat{F}(\omega) = \sum_{n=-\infty}^{+\infty} f[n]w[n]e^{-j\omega n} \tag{3.18}$$

where w[n] = a weighting vector. Notice the summation now extends over an infinite number of samples.

Equation 3.18 explicitly shows the parameters the engineer controls in the estimation process. The underlying random process and sampling characteristics completely control f[n], allowing external influence only when employing an active source. The exponential kernel is an inherent characteristic of the Fourier transform. Other kernels may be considered, but for spectrum estimation involving complex exponentials, the Fourier transform exhibits the greatest applicability. Therefore, the weight vector w[n] represents the only parameters the engineer exhibits complete control over. In fact, the weight vector and sampling characteristics completely determine the spectral filter characteristics.

3.5.3 <u>The Spectral Smoothing Kernel</u>

The DTFT of the weight vector w[n]

$$W(\omega) = \sum_{n=-\infty}^{+\infty} w[n]e^{-j\omega n} \tag{3.19}$$

describes the spectral smoothing characteristics of the filter. The estimated spectrum equals the true spectrum convolved with the smoothing kernal W(ω).

The weight vector controls the filter mainlobe and sidelobe characteristics. Two commonly encountered windows are the rectangular and Bartlett windows. Figure 3.5 shows two discrete time windows and their corresponding DTFTs. The rectangular smoothing function exhibits a narrower mainlobe width, but the Bartlett smoothing function exhibits much greater sidelobe control. The two windows show that the actual optimum filter for a given process depends on the desired estimation parameters. In the examples shown, the rectangular window shows greater resolution and amplitude estimation capabilities, while the Bartlett window shows greater ability to control wide band noise and leakage from competing signals.

The most important characteristics of the smoothing function are sidelobe height and mainlobe width, which control leakage of energy and resolution, respectively. A typical measure of resolution is the Rayleigh criterion (half of the main lobe width for symmetric smoothing functions) or the full-width half-maximum (which measures the width at the middle height of the mainlobe) (Johnson and Dudgeon, 1993). Another problem related to resolution depends on the sidelobe height. If a sidelobe is large, it will mask very small signals found at remote portions of the spectrum. The sidelobes control how portions of the spectrum away from the mainlobe affect the power estimate at the frequency of interest. The mainlobe and sidelobe structure can be modified with different choices of window weighting functions and sampling characteristics. A longer total length of data N for a given t_s decreases mainlobe width, but a longer t_s for a given length N will decrease the bandwidth available for power estimation.

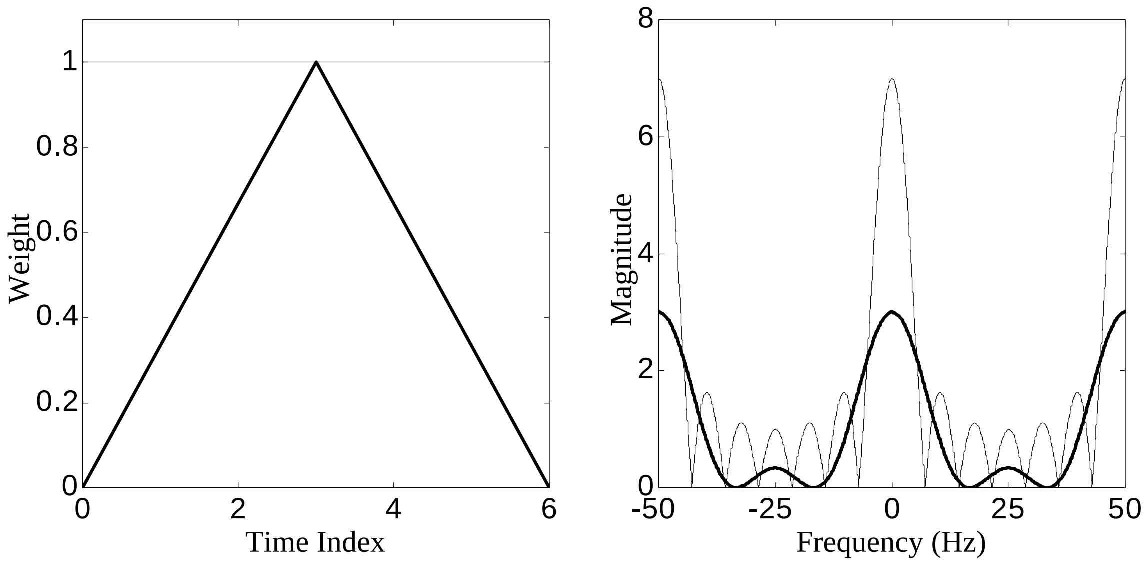

Figure 3. 5 Window Choice and Spectral Properties. The rectangular (light line) and Bartlett (bold line) windows are shown in the left panel, and the magnitude of their Discrete Time Fourier Transforms are shown in the right panel for number of samples N = 7 and a sampling frequency of 50 Hz.

3.6 Power Spectrum Estimators

A stationary random process can be characterized by a power spectral density function (Capon, 1969). The power spectrum and the autocorrelation function are Fourier transform pairs. Obtaining a statistically good estimate of the power in a random process is the overriding goal of power spectral density (PSD) estimation. Acceptable estimates rely upon appropriate data properties as well as proper choice of window weighting functions.

The following discussion is necessarily brief, with only the concepts important to the subsequent discussion of multidimensional spectrum estimation being presented. The one-dimensional frequency spectrum estimation problem controls resolution of frequency in the multidimensional power estimation problem. The PSD estimation problem requires designing a one-dimensional spectral filter with optimum capability of isolating a single frequency as shown in Equation 3.15. Adjusting the weighting function and the sampling characteristics allows problem specific optimization of the spectral filter properties.

If a rectangular window is applied to the data vector in the temporal domain, the window weight vector applied to the autocorrelation function corresponds to a rectangular window convolved with itself, forming a Bartlett window. Therefore, the actual smoothing kernel in the power spectral domain becomes a $(\text{sinc})^2$ function when using a rectangular weight vector. As discussed previously, the overriding goals of spectrum estimation are sidelobe leakage control and mainlobe width. This chapter will introduce basic spectrum estimation techniques, delaying discussion of more advanced methods until Chapter 4, which will deal with multidimensional spectrum estimation problems.

3.6.1 Fourier Spectra of Deterministic and Repeatable Signals

Although most naturally occurring processes produce random signals, in some engineering applications, signals may be deterministic. For example, impulse response tests utilizing a repeatable source on a linear shift invariant system may be stacked in the time domain to remove noise. If the noise is statistically stationary with a zero mean, stacking will reduce the noise variance. In the limit, if all noise may be removed, the resulting signal may be considered a deterministic realization of the system impulse response. In active SASW tests, experimental measurements may be stacked in the temporal domain to reduce background seismic noise.

The advantage of deterministic signals lies in the ability to exactly characterize the spectral components with the DTFT. If noise remains in the temporal signals, assumptions about the noise statistics must be made, and the resulting spectrum only approaches the true process spectrum. Stacking in the frequency domain will only reduce the variance of the frequency power content estimates, without affecting noise power.

3.6.2 Periodogram

If separate realizations of a process cannot be stacked without destroying the underlying signal information, power spectrum estimation requires additional methods and theoretical development. An early power spectrum estimator for random processes is the periodogram. The periodogram is the Fourier transform of the autocorrelation sequence $r_s(\ell)$, producing the estimated power

$$P_{periodogram}(\omega) = \sum_{\ell=-(N-1)}^{N-1} r_s(\ell)\exp(-j\ell\omega) \tag{3.20}$$

where $P_{periodogram}(\omega)$ = the estimated power in the frequency ω. Using the convolution theorem and the Fourier transform of the data s(n) directly, the periodogram can be expressed as (Hayes, 1996)

$$P_{periodogram}(\omega) = \frac{1}{N} S(\omega)S^*(\omega) = \frac{1}{N}\left|S(\omega)\right|^2 \tag{3.21}$$

where $S(\omega)$ = the Fourier transform of the data vector s(n) and the * indicates complex conjugation. The periodogram can be interpreted as a bank of bandpass or "matched frequency" filters, each centered at a desired frequency, with the passband width equal to the mainlobe of the smoothing function. The periodogram is an early PSD estimation procedure, and is easy to compute, but it exhibits limited accuracy, especially for short data records (Hayes, 1996). The left panel of Figure 3.6 shows an overlay plot of 10 periodograms of a single complex exponential in white Gaussian noise.

3.6.3 Bartlett's Procedure

Averaging of several periodograms, or Bartlett's method, produces a better power spectrum estimate in terms of statistical characteristics. The power estimate is given by

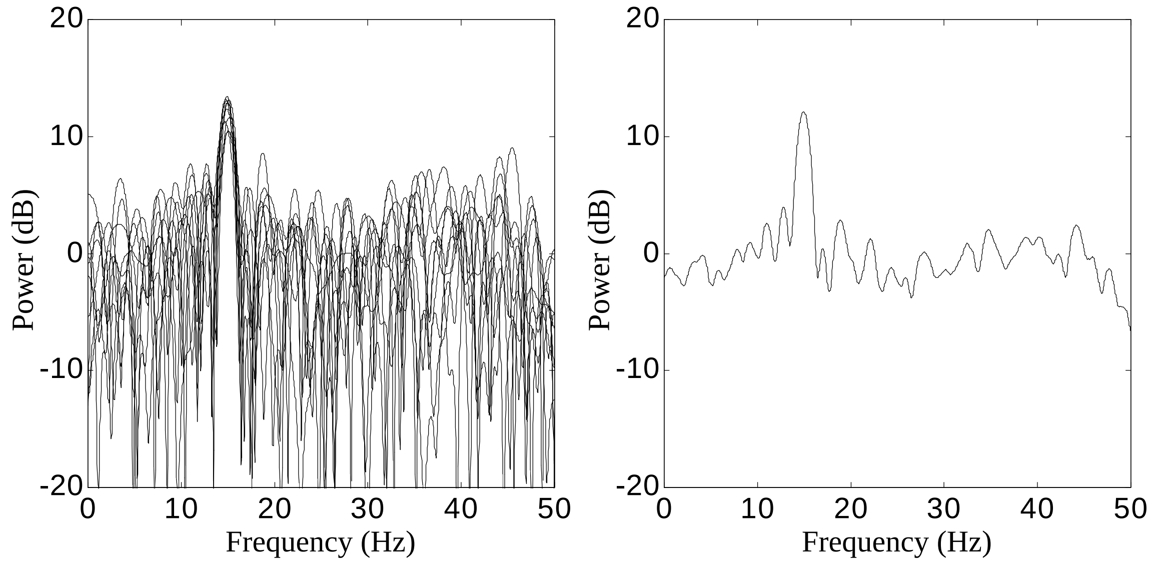

Figure 3. 6 Periodograms and Periodogram Averaging. Left Panel: Overlay of ten periodogram estimates; Right Panel: Average of the 10 periodograms in the left panel.

$$P_{\text{Bartlett}}(\omega) = \frac{1}{B}\sum_{i=1}^{B} P_{\text{periodogram}}^{i}(\omega) = \frac{1}{B}\sum_{i=1}^{B}\frac{1}{L} S_i(\omega) S_i^*(\omega) \tag{3.22}$$

where B = number of blocks of data to be averaged, $P_{\text{periodogram}}^{i}(\omega)$ = the periodogram estimate from the i^{th} block, $S_i(\omega)$ = the frequency spectrum from the i^{th} block, and the block length L = N/B. Averaging multiple periodograms sacrifices frequency resolution in exchange for a reduction in the variance of the power spectrum estimate (Hayes, 1996). The right panel of Figure 3.6 shows the Bartlett method estimate obtained from averaging the 10 periodogram estimates shown in the left panel of Figure 3.6.

3.6.4 Other Estimators

Using windows other than the rectangular window in the periodogram estimate allow a tradeoff between resolution and sidelobe height. Welch's method refers to the process of averaging periodograms modified with a window weight vector. For fixed length data sequences, the autocorrelation estimates at large lags receive the least number of samples, and therefore, have the least reliability. Windowing the autocorrelation sequence applies smaller weights to the larger lags, decreasing their contribution in the periodogram estimate.

3.7 Advanced Spectrum Estimation Techniques

Several additional spectrum estimation techniques are available. The techniques vary depending on the model and data correlation assumptions. For example, the FFT periodically replicates the sampled sequence outside the length from 0 to N-1. In many cases, this will not be the optimal data model. Additionally, all the previous spectrum estimation techniques use a fixed filter structure for estimation, i.e. a fixed mainlobe and sidelobe structure. Adaptive techniques actually change the filter structure depending on the observed signal and noise characteristics, allowing large gains in signal to noise ratios and resolution in many cases. Dynamic adaptive techniques represent algorithms that not only update the filter depending on correlation estimates, but also update the correlation estimates during measurements (Johnson and Dudgeon, 1993). Although many of the techniques are applicable to both the one-dimensional and multidimensional problems, a complete discussion of the methods will be delayed until the end of Chapter 4, when the full power of multidimensional vector notation may be utilized.

3.8 Extension to Multidimensional Problem

The multidimensional estimation problems encountered in Chapter 4 will bear striking similarities to the one-dimensional problems discussed in this chapter. In many cases, the power spectrum estimate equations and methods are exactly analogous, except for the extension to matrices, vectors, and multidimensional problem constraints. The vector notation actually streamlines the implementation of equations and aids in the analysis of random processes as an indexed sequence of random variables. Although the multidimensional problem contains spatial and temporal variables, the one-dimensional estimation methods covered in this chapter will always control frequency resolution estimation from temporal domain measurements. The multidimensional problem will introduce several complications and extensions, but the added analytical power offsets the added complexity, allowing a significant increase in information and improvement in statistical properties of parameter estimates.

3.9 Summary

Fourier theory yields many tools for the analysis of experimentally measured signals. Fourier series represent an ideal rarely found in nature, and therefore, the Fourier transform is most commonly used for signal examination. Several different forms of the Fourier transform exist, and although the DFT is the most common form used in practice, the DTFT yields considerable insight into underlying spectral operations.

Temporal domain power spectrum estimation relies on acceptable estimation of the autocorrelation function. The choice of sampling characteristics and weighting functions significantly impacts the ability to resolve and identify spectral components. The longest sampled temporal lag controls resolution, and the sampling rate controls the bandwidth available for energy content estimation. The weighting function exchanges decreases in sidelobe height, which controls energy leakage and the ability to identify relatively small signals that may be masked by large sidelobes, for reductions in resolution. The periodogram is an early power spectrum estimation technique, and Bartlett's method decreases the variance of the power spectrum estimate by averaging periodograms. Advanced spectrum estimation techniques are introduced in Chapter 4.