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Love's formula and *H/V*-ratio (ellipticity) of Rayleigh waves

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Abstract

The ellipticity of Rayleigh surface waves, which is an important parameter characterizing the propagation medium, is studied for several models with increasing complexity. While the main focus lies on theory, practical implications of the use of the horizontal to vertical component ratio (*H/V*-ratio) to study the subsurface structure are considered as well. Love's approximation of the ellipticity for an incompressible layer over an incompressible half-space is critically discussed especially concerning its applicability for different impedance contrasts. The main result is an analytical exact formula of *H/V* for a 2-layer model of compressible media, which is a generalization of Love's formula. It turns out that for a limited range of models Love's approximation can be used also in the general case.

1. Introduction

Rayleigh waves propagating over the surface of homogeneous and inhomogeneous elastic half-spaces are a well-known and prominent feature of wave theory. They are vector waves, which are confined to the region near the surface, and are polarized in the saggital plane. That means, the components of displacement are a horizontal component, which is parallel to the direction of propagation, and a vertical component directed into the half-space. The dimensionless ratio of these components *H/V* at the surface, the so-called ellipticity, is an important parameter which reflects fundamental properties of the elastic material.

Indirectly, the study of Rayleigh wave ellipticities has recently gained considerable popularity in the context of studying ambient seismic vibrations for seismic hazard analysis. Since ambient vibrations as generated by wind, traffic, etc. consist predominantly of surface waves (Correig and Urquizu [1]; Douze [2]; Ohmachi and Umezono [3]), *H/V* power spectral ratios of ambient vibrations provide a statistical means to look at Rayleigh wave ellipticities. As a consequence, *H/V* spectral ratios of ambient vibrations are increasingly used for the investigation of local site amplification during strong earthquakes (Bard [4]; Kudo [5]). Due to the strong impedance contrast in the shallow subsurface structure, local site effects are often fairly well predicted by simple models (Ohrnberger et al. [6]; Scherbaum et al.[7]). Therefore, a thorough theoretical understanding of even a single layer over halfspace is not only of theoretical but also of considerable practical interest. Adding to this argument is the fact that an accepted theoretical model for the interpretation of *H/V* measurements from ambient vibrations, still has to be developped. Furthermore, the *H/V*-ratio has recently also found practical applications in global seismology (Munirova and Yanovskaya [8]) and was proposed to use in non-destructive testing with acoustic surface waves by Malischewsky et al. [9].

It is well-known but still remarkable that for an homogeneous half-space *H/V* can be expressed by a very simple formula. Adding only a single layer, immediately complicates the situation considerably. To our knowledge, only very few studies deal with the attempt to derive formulas for this case, among them the famous thesis of Love [10]. His derivation deals with an incompressible layer over an incompressible half-space for which he presented

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an extremely simple approximation for *H/V*. The range of application of this formula, however, remained unclear. In order to better understand the properties of *H/V* in a simple, but still practically relevant situation, we have generalized Love's argument for compressible media and present it in this paper in a modern notation. The result is an exact explicite formula of *H/V* for the general case of one layer over a half-space. It turns out that Love's approximation, originally derived for incompressible media, may be applied for compressible media as well but is valid only in a limited range of cases. The paper is structured such that the entity *H/V* is discussed for models of increasing complexity: homogeneous half-space, impedance surface, layer over half-space.

2. The homogeneous half-space

Although the following calculations are straightforward and well documented in the textbook literature (e. g. Ben-Menahem and Singh [11]), we felt it to be useful for the understanding of the more complicated models to briefly present the general ideas to express *H/V* for this situation as well. The 2D-Rayleigh wave motion is described in a cartesian coordinate system with its origin located on the surface of the half-space. The x_1- axis points into the direction of propagation while the x_3 - axis is directed into the half-space. Our starting point is the Navier equation

$$\mu \frac{\partial^2 u_i}{\partial x_j \, \partial x_j} + (\lambda + \mu) \frac{\partial^2 u_j}{\partial x_i \, \partial x_j} = \rho \, \ddot{u}_i \;\; , \;\; i = 1, 2, 3 \;\; , \qquad (1)$$

with the components of the displacement vector denoted by u_i, λ and μ are Lamé's parameters, and ρ is density. Einstein's summation condition is understood and the time derivative is denoted by a dot. The depth-dependent Rayleigh eigenfunctions are $U_i = U_i(x_3)$ $(i = 1, 3)$. The assumption of harmonic plane waves

$$u_i = U_i(x_3) \, \mathrm{e}^{\mathrm{i}(k x_1 - \omega t)} \;\; , \;\; i = 1, 3 \;\; , \qquad (2)$$

with wave number k, angular frequency ω, and time t leads to the following coupled system of differential equations of second order:

$$\begin{aligned} \gamma \, U_1''(x_3) \; + \; \mathrm{i} k (1\text{-}\gamma) \, U_3'(x_3) \; - \; P \, U_1(x_3) \; &= \; 0 \;\; , \\ U_3''(x_3) \; + \; \mathrm{i} k \, (1-\gamma) \, U_1'(x_3) \; - \; \gamma \, Q \, U_3(x_3) &= \; 0 \;\; . \end{aligned} \qquad (3)$$

The derivatives with respect to x_3 are labeled by dashes. The imaginary unit is denoted by i, γ is the squared ratio of shear-wave velocity β to longitudinal-wave velocity α,

$$\gamma = \frac{\beta^2}{\alpha^2} = \frac{\mu}{\lambda + 2\mu} \;\; , \qquad (4)$$

and P and Q are defined by

$$P = k^2 - k_\alpha{}^2 \;\; , \;\; Q = k^2 - k_\beta{}^2 \;\; , \qquad (5)$$

where k_α and k_β are the wave numbers of longitudinal and transversal waves, respectively. By introducing the square roots of P and Q,

$$p = \sqrt{P}\ ,\ \ q = \sqrt{Q}\ , \tag{6}$$

and the integration constants C_1, C_2, C_3, C_4, the general solution of (3) can be written as:

$$\begin{aligned} U_1(x_3) &= C_1\,\mathrm{e}^{-p x_3} + C_2\,\mathrm{e}^{p x_3} + C_3\,\mathrm{e}^{-q x_3} + C_4\,\mathrm{e}^{q x_3}\,, \\ U_3(x_3) &= \mathrm{i}\left[\frac{p}{k}\,C_1\,\mathrm{e}^{-p x_3} - \frac{p}{k}\,C_2\,\mathrm{e}^{p x_3} + \frac{k}{q}\,C_3\,\mathrm{e}^{-q x_3} - \frac{k}{q}\,C_4\,\mathrm{e}^{q x_3}\right]. \end{aligned} \tag{7}$$

For the half-space $C_2 = C_4 \equiv 0$ must hold. The remaining constants C_1 and C_3 are usually determined from the condition of a stress-free surface

$$S_{i3} = 0,\ \ i = 1, 3\ \text{ for } x_3 = 0, \tag{8}$$

where $S_{i3}(x_3)$ are the corresponding x_3- dependent stress tensor components defined by

$$\begin{aligned} S_{13}(x_3) &= \rho\,\beta^2\,[U_1'(x_3) + \mathrm{i}\,k\,U_3(x_3)], \\ S_{33}(x_3) &= \rho\,\alpha^2\,[U_3'(x_3) + \mathrm{i}\,k\,(1\text{-}2\gamma)\,U_1(x_3)]\,. \end{aligned} \tag{9}$$

Setting the determinant of the homogeneous system (8) for C_1 and C_3 to zero results in Rayleigh's equation with the phase velocity $c = \omega / k$ and $\xi = c/\beta$:

$$\begin{aligned} &4\,p\,q - k^2\left(2 - \frac{c^2}{\beta^2}\right)^2 = 0 \quad \text{or} \\ &F(\xi) = 4\,\sqrt{1 - \gamma\,\xi^2}\ \sqrt{1 - \xi^2} - (2 - \xi^2) = 0 \end{aligned} \tag{10}$$

and

$$C_3 = -\,\frac{2\,p\,q}{Q + k^2}\,C_1\ . \tag{11}$$

The simple formula for *H/V* mentioned above is then (see e. g. Ben-Menahem and Singh [11]):

$$\chi = |H/V| = |U_1(0)/U_3(0)| = \sqrt{q/p} = 2\,\frac{\sqrt{1 - c^2/\beta^2}}{2 - c^2/\beta^2}\ . \tag{12}$$

The ellipticity χ depends only on Poisson's ratio ν. In terms of the phase velocity c, it is expressed here for the first time analytically by applying the formula of Malischewsky [12]. With the auxiliary functions h_1, h_2, h_3, h_4, defined by

$$h_1(\nu) = \sqrt{\frac{5 - 21\nu + 16\nu^2 - 32\nu^3}{(\nu - 1)^3}}\ ,\ h_2(\nu) = \frac{11 - 56\nu}{\nu - 1}\ ,\ h_3(\nu) = \sqrt[3]{4}\ \sqrt[3]{3\sqrt{3}\, h_1(\nu) + h_2(\nu)}\ ,$$

$$h_4(\nu) = \sqrt[3]{4}\ \mathrm{sign}(2 - 5\nu)\ \sqrt[3]{[-3\sqrt{3}\ h_1(\nu) + h_2(\nu)]\ \mathrm{sign}(2 - 5\nu)}\ , \tag{13}$$

we obtain

$$\chi = 2\sqrt{3}\ \frac{\sqrt{h_3(\nu) + h_4(\nu) - 5}}{h_3(\nu) + h_4(\nu) - 2}\ . \tag{14}$$

The symbol sign(x) stands for the signum function. It is assumed that the cubic root is located in the first and fourth quadrant, depending on the sign of the imaginary part in the argument of the root. Fig. 1 shows the well-known behaviour of χ in dependence on ν for all possible values of Poisson's ratio. It should be noted that, contrary to the models to be discussed in the following, there is no dependence on frequency.

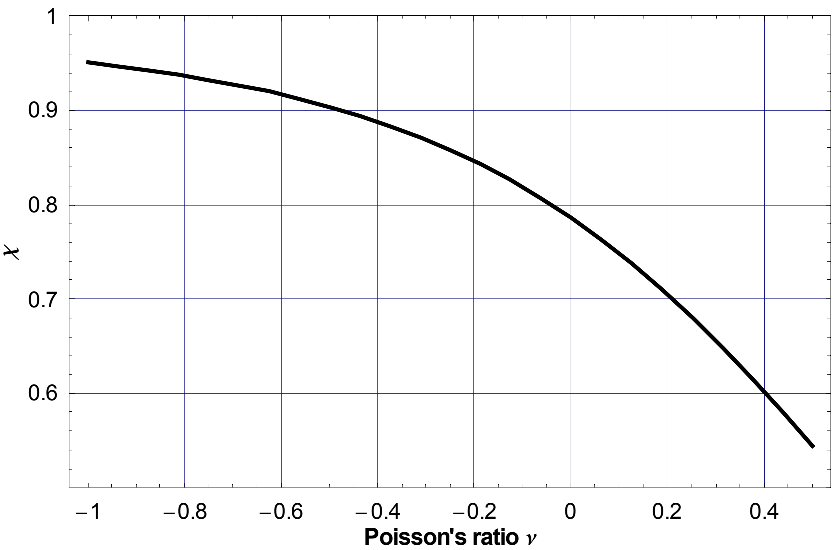

Fig. 1: The ellipticity χ for the homogeneous halfspace in dependence on Poisson's ratio ν

3. Impedance surface

In a low frequency approximation, Tiersten [13] introduced special boundary conditions on the surface in order to simulate the elastic behaviour of a thin layer over an half-space. Fig. 2 shows the assumed configuration. Note that in this case the origin of the coordinate system is located on the boundary between layer and half-space.

The elastic parameters in the layer are indexed as 1 and unindexed for the half-space, respectively. The thickness of the layer is *d*. For Rayleigh-wave motion the stress-free conditions (8) are replaced by Tiersten's boundary conditions

$$S_{13} + \varepsilon_1 U_1 = 0,\ S_{33} + \varepsilon_3 U_3 = 0 \quad \text{for } x_3 = 0, \tag{15}$$

with

$$\varepsilon_1 = d\,\rho_1\,\omega^2\left[1 - \frac{4\,(\mu_1+\lambda_1)}{(2\,\mu_1+\lambda_1)}\,\frac{\beta_1{}^2}{c^2}\right] \text{ and } \varepsilon_3 = d\,\rho_1\,\omega^2 . \tag{16}$$

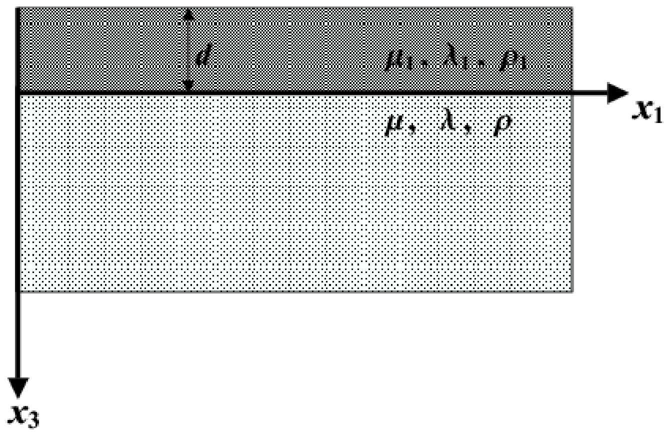

Fig. 2: Layer with thickness *d* over a halfspace

Recently Bövik [14] suceeded in improving these boundary conditions by introducing derivatives of stress components on the right sides of (15). They are then correct in an asymptotic sense up to the order *O*(*d*) (so-called *O*(*d*) – boundary conditions). A further discussion of the implications of both kinds of special boundary conditions is beyond the scope of this article. Here we calculate the ellipticity of Rayleigh waves under the conditions (15). The general solution is the same as in (7). But in applying (15), we realize that Rayleigh's equation (10) has to be replaced by the frequency-dependent equation (compare with Malischewsky [15])

$$\omega^2 F(\xi) - \frac{\omega\,\beta}{\mu}\left(\varepsilon_3\sqrt{1-\gamma\,\xi^2} + \varepsilon_1\sqrt{1-\xi^2}\right)\xi^3 + \frac{\varepsilon_1\,\varepsilon_3}{\mu^2}\beta^2\,\xi^2\left(1-\sqrt{1-\gamma\,\xi^2}\,\sqrt{1-\xi^2}\right) = 0 \tag{17}$$

and (11) becomes

$$C_3 = -\frac{q\,(\varepsilon_1\,p - 2\,\rho\,\beta^2 P)}{p[\varepsilon_1\,q - \rho\,\beta^2\,(k^2+Q)]}\,C_1 \ . \tag{18}$$

After some algebra, the ellipticity of Rayleigh waves for the impedance surface can be written as

$$\chi = \frac{k\,\rho\,\beta^2\,[p\,k^2 + q\,(p\,q - 2\,P)]}{\varepsilon_1\,(q\,P - p\,k^2) + \rho\,\beta^2 k_\beta{}^2\,P} \ . \tag{19}$$

Expression (19) for the ellipticity for model 1 (see Table 1) is presented in Fig. 3 as a function of the dimensionless parameter d/λ_{β_1} with the wavelength λ_{β_1} of the shear waves in the layer. Here, the ellipticity for the homogeneous halfspace with stress-free boundary conditions is additionally included as a dashed line. It becomes obvious that the introduction of the simple impedance-surface model already yields a strong frequency dependence of the ellipticity. However, the peak appears at lower frequencies with respect to d/λ_{β_1} than is often observed for realistic sedimentary site models where the peak is close to d/λ_{β_1} =0.25 (e.g. Scherbaum et al. [7]; see also discussion related to Fig. 5).

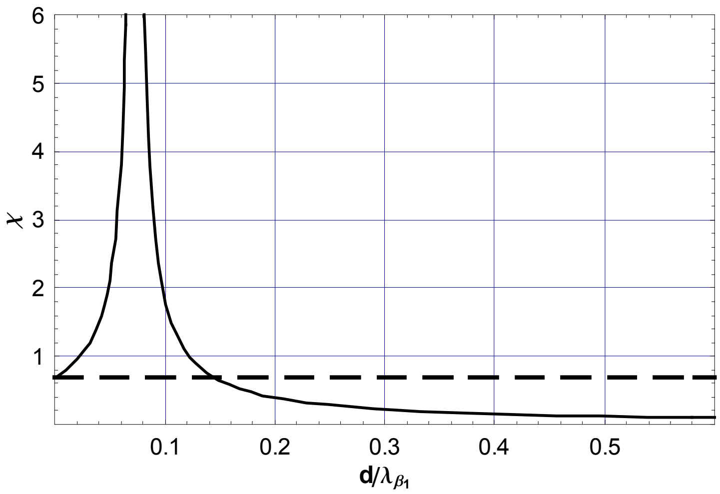

Fig. 3: Ellipticity χ of the impedance surface (full) and the halfspace (dashed) in dependence on d / λ_{β_1}

4. Layer over half-space

In the following we consider the most interesting case of a not necessarily thin layer over an half-space. The geometry is as in Fig. 2 and we use label 1 for the parameters of the layer and label 2 for the half-space, respectively. Love [10] investigated this problem under the simplifying assumption that both media are incompressible. We do not reproduce Love's original derivation in detail here, but in the course of its generalization for compressible media we follow his argumentation by and large. Since Love's approach is not very well-known, it is worth to make a few remarks about the incompressible case, though. In this case, Lamé's parameter λ and the velocity of longitudinal waves α are infinite. In formulating the equation of motion it has to be taken into account that the product

$$\lambda \theta = \lambda \, u_{i,i} = \Pi \quad , \tag{20}$$

where θ is the vanishing volume strain, adopts a finite value Π, which is interpreted by Love as a hydrostatic pressure. In addition, the stress component S_{33} has to be modified in the same manner. The modified equation of motion can be solved by introducing scalar and vector potentials and prescribing a convenient value for Π. On the other hand, the general solution (7) and the period equation (10), respectively, are also valid for the incompressible case when taking the limit $\alpha \to \infty$. The root of Rayleigh's equation for incompressible media was presented analytically by Malischewsky [16]. We continue with the compressible case by writing the solutions for the layer and the half-space in a modified way in order to be more consistent with Love:

$$U_1^{(1)}(x_3) = \mathrm{i}\left[-L_1 \cosh(p_1 x_3) + L_2 \sinh(p_1 x_3) - L_3 \cosh(q_1 x_3) + L_4 \sinh(q_1 x_3)\right],$$
$$U_3^{(1)}(x_3) = \frac{p_1}{k}\left[-L_1 \sinh(p_1 x_3) + L_2 \cosh(p_1 x_3)\right] + \frac{k}{q_1}\left[-L_3 \sinh(q_1 x_3) + L_4 \cosh(q_1 x_3)\right],$$
$$U_1^{(2)}(x_3) = -\mathrm{i}\left[A_1 \mathrm{e}^{-p_2 x_3} + A_2 \mathrm{e}^{-q_2 x_3}\right],$$
$$U_3^{(2)}(x_3) = A_1 \frac{p_2}{k} \mathrm{e}^{-p_2 x_3} + A_2 \frac{k}{q_2} \mathrm{e}^{-q_2 x_3} \ . \tag{21}$$

Here L_1, L_2, L_3, L_4 are the integration constants for the layer and A_1, A_2 for the half-space, respectively. The relevant stress tensor components, belonging to these eigenfunctions, are denoted by $S_{13}^{(1)}(x_3), S_{33}^{(1)}(x_3); S_{13}^{(2)}(x_3), S_{33}^{(2)}(x_3)$. In the following, because of the cumbersome algebra we are omitting some of the intermediate results in detail and focus on the essential steps and the final result. The stress-free conditions of the surface

$$S_{13}^{(1)}(-d) = S_{33}^{(1)}(-d) = 0 \tag{22}$$

together with the continuity relations on the boundary between the layer and the half-space

$$U_1^{(1)}(0) = U_1^{(2)}(0),\ U_3^{(1)}(0) = U_3^{(2)}(0),\ S_{13}^{(1)}(0) = S_{13}^{(2)}(0),\ S_{33}^{(1)}(0) = S_{33}^{(2)}(0) \tag{23}$$

yield an homogeneous system of 6 equations for the 6 constants $L_1, L_2, L_3, L_4, A_1, A_2$. Its determinant has to be zero to yield the period or secular equation for this model. This equation determines the phase velocity c of Rayleigh waves in terms of the frequency or the wave length. There are several possibilities to write this complicated equation, which is a generalization of (10) and (17), in a convenient manner. We used the formula of Ben-Menahem and Singh [11], which is given here only symbolically as

$$\Delta\,(c,\omega) = 0 \ . \tag{24}$$

This equation depends on 8 parameters: 6 elastic parameters, layer's thickness and frequency. It is nor surprising that it is impossible to discuss the roots of this equation in complete generality. Instead we pick out some typical parameter combinations, which are important for practical reasons. The same is true for the corresponding *H/V*-ratio which will be discussed in the same manner.
Let us assume that (24) is solved already. The crucial trick of Love in order to get a reasonable analytical expression for the ellipticity was to express the constants $L_1 - L_4$ by A_1 and A_2 by applying the continuity relations (23)

$$\begin{aligned} L_1 &= l_{11} A_1 + l_{12} A_2\,,\ L_2 = l_{21} A_1 + l_{22} A_2\,, \\ L_3 &= l_{31} A_1 + l_{32} A_2\,,\ L_4 = l_{41} A_1 + l_{42} A_2\,, \end{aligned} \tag{25}$$

where the coefficients $l_{11} - l_{42}$ are complicated functions of the 8 parameters mentioned above. Furthermore, it is possible to introduce these equations into the stress-free conditions (22) which yields the two equations

$$c_{11} A_1 + c_{12} A_2 = 0,\ c_{21} A_1 + c_{22} A_2 = 0, \tag{26}$$

where the coefficients $c_{11} - c_{22}$ are again complicated functions of the 8 parameters. In order to obtain the ellipticity χ it is necessary to form the expressions $U_1^{(1)}(-d)$ and $U_3^{(1)}(-d)$ from (21). By using (25) these expressions are linear functions of the half-space constants A_1 and A_2. It turns out that they can be considerably simplified by introducing the relations (26). After this step the ellipticity χ is written as

$$\chi = \left| \frac{U_1^{(1)}(-d)}{U_3^{(1)}(-d)} \right| = \frac{d_{11} A_1 + d_{12} A_2}{d_{21} A_1 + d_{22} A_2} \tag{27}$$

with the new coefficients $d_{11} - d_{22}$. We are able to eliminate A_2 by using e. g. the first equation (26) and obtain

$$\chi = \frac{c_{12}\, d_{11} - c_{11}\, d_{12}}{c_{12}\, d_{21} - c_{11}\, d_{22}} \quad . \tag{28}$$

Finally, it is convenient to write this expression as a product of 3 factors f_1, f_2, f_3 :

$$\chi = f_1 \cdot f_2 \cdot f_3 \ ,$$

$$f_1 = 1 - \frac{c^2}{2\beta_1^{\,2}} \ , \ f_2 = \frac{1}{\sqrt{1 - c^2/\alpha_1^{\,2}}} \ , \ f_3 = \frac{1 + y \tanh(d\, p_1)}{y + \tanh(d\, p_1)} \quad . \tag{29}$$

The entity y is a very complicated function of the 8 parameters mentioned above and is presented in the appendix. It should be noted that this final result can be obtained in a reasonable manner only by using symbolic calculation as in MATHEMATICA. It is valid also for all higher modes of Rayleigh waves, but in the following we will only discuss the fundamental mode. Love's result for incompressible media was

$$\chi_{inc} = f_1 \cdot \tilde{y} = \left(1 - \frac{c^2}{2\,\beta_1^{\,2}} \right) \cdot \tilde{y} \quad , \tag{30}$$

where $\tilde{y}$ has a similar structure as our factor f_3, but it is not identical. Love's argumentation was that $\tilde{y}$ is a slowly varying function very near to 1 so that the simple formula

$$\chi_{inc} \approx 1 - \frac{c^2}{2\,\beta_1^{\,2}} \tag{31}$$

is a good approximation for $|H/V|$. However, it should be critically noted that by far not all parameter combinations yield such a simple result as we will see later.
Next we will discuss the general case of compressible media. It can be easily realized from (29) that for the limit $\omega \to \infty$ f_3 is unity and $f_1 \cdot f_2$ adopts the half-space value χ according to (12) with the parameters of the layer. Because of the complexity of y, the value for the other limit $\omega \to 0$ cannot be deduced analytically in a simple manner, but numerically it

approximately adopts the half-space value (12) with the parameters of the half-space. It remains an open question here, wether the eigenfunctions of an half-space with a very thin layer change exactly or only approximately into the half-space eigenfunctions in the limit $\omega \to 0$. For intermediate ω-values between 0 and infinity, the exact formula (29) has to be used. It turns out, however, that for certain parameter regions the product $f_2 \cdot f_3$ is fairly close to 1, so that in these - and only in these - special cases Love's formula (31) can be used for compressible media as well. To demonstrate this we use two different models with high {1} and low {2} shear wave contrasts, for which the parameters are given in Table 1.

	parameters	model 1	model 2
layer	α_1 [km/sec]	1.5000	3.0000
	β_1 [km/sec]	0.5000	1.0000
	ρ_1 [g/cm^3]	2.0000	2.0000
	ν_1	0.4375	0.4375
	d [km]	0.3000	0.3000
half-space	α_2 [km/sec]	5.2000	5.2000
	β_2 [km/sec]	3.0000	3.0000
	ρ_2 [g/cm^3]	2.7000	2.7000
	ν_2	0.2506	0.2506

Tab. 1: Parameters for model 1 and model 2

In Fig. 4, the product $f_2 \cdot f_3$ is presented for both models as a function of d/λ_{β_1}. We realize that only for model 1 with the higher velocity contrast there is an intermediate region of d/λ_{β_1} - values (between 0.25 and 0.45), where $f_2 \cdot f_3$ approximates 1, while for both models the product goes to 1 for $d/\lambda_{\beta_1} > 0.55$.

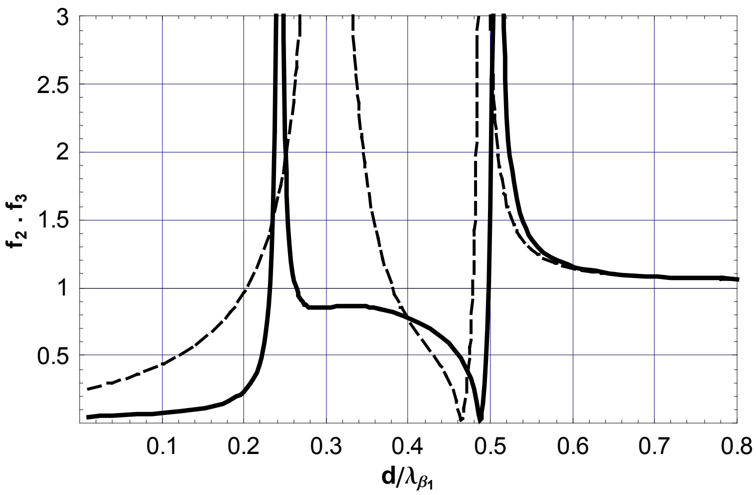

Fig. 4: The product $f_2 \cdot f_3$ for model 1 (full) and model 2 (dashed), respectively, in dependence on d / λ_{β_1}

In Fig. 5 , we present a comparison between the exact ellipticity after (29) and the approximation (31) as a function of d/λ_{β_1} .

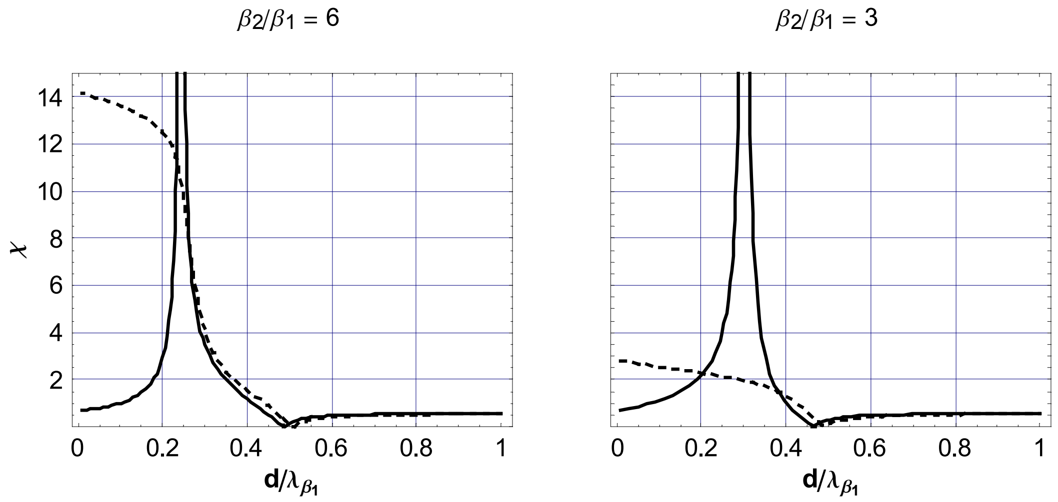

Fig. 5: Exact values of χ for model 1 (left, full) and model 2 (right, full) and Love's approximation for model 1 (left, dotted) and model 2 (right, dotted) in dependence on d/λ_{β_1}

For higher d/λ_{β_1} - values (> 0.5), the coincidence is very good for both models. However, the range below the root of χ, which is very important for practical applications, is much better approximated by Love's formula for model 1 with the higher beta contrast than for model 2. It may be that this is a general tendency. The singularity of *H/V* requires special considerations. For practical applications is is often assumed that the corresponding frequency is related to the so-called "shear-wave resonance" in the layer (see e. g. Mooney and Bolt [17]; Nakamura[18]; Muciarelli[19]) . That means it is assumed that the singularity occurs for such frequencies, where the layer's thickness is one quarter of the wavelength of shear waves within the layer, i. e. by using our nomenclature d/λ_{β_1} = 0.25. Until now, this statement was never proven analytically. We realize from Fig. 5, that it is very well fulfilled for model 1, but only approximately for model 2. Due to the enormous practical consequences, it seems worthwhile to investigate the validity of this statement for different shear-wave contrasts by using our exact formula (29). Fig. 6 shows d/λ_{β_1} for the peak value of *H/V* versus the beta contrast β_2/β_1. Here, the parameters of the half-space are not changed and Poisson's ratio of the layer is assumed to be constant as in Tab. 1.

It is remarkable that for beta contrasts greater than 3.5 the statement is very well fulfilled which is in conformity with practical experience from site conditions with high impedance contrast (Bard[20]). But it cannot be said, that the statement is generally true. Especially, the occurence of a strange singularity for $\beta_2/\beta_1 \approx 2.6$ requires an additional investigation, which ,however, is beyond the scope of the present paper.

It is also interesting to study the influence of Poisson's ratio ν_1 in the layer on the quality of the approximation. The following Fig. 7 shows a 2D-density plot of the standard deviation δ in percent of Love's approximation for two models with higher and lower shear-wave contrast, respectively, in dependence on ν_1 and d/λ_{β_1} .

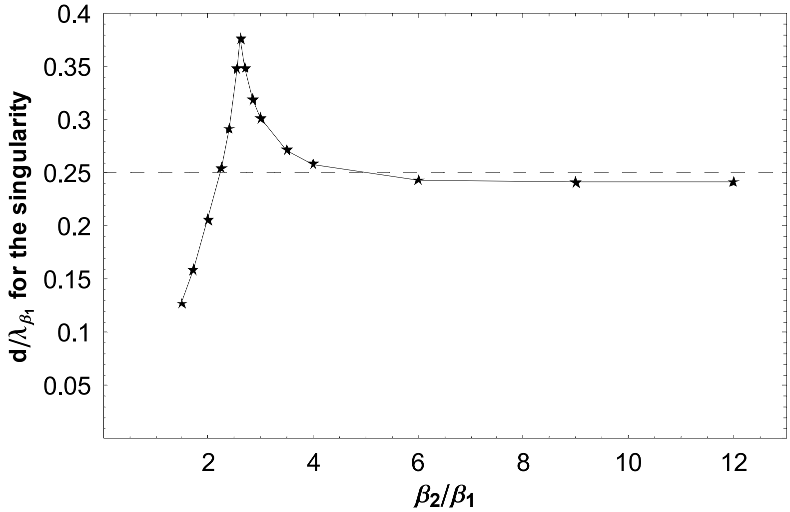

Fig. 6: d/λ_{β_1} for the peak value of *H/V* in dependence of β_2/β_1

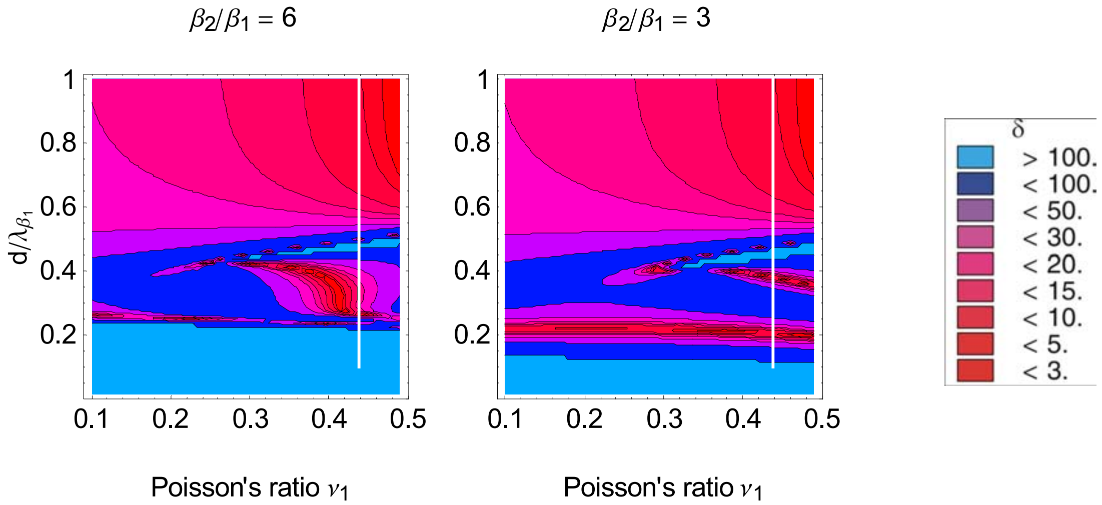

Fig.7: Contour plot of the standard deviation δ in percent between exact χ and Love's approximation for model 1 (left) and model 2 (right) in dependence on ν_1 and d/λ_{β_1}. The white lines indicate $\nu_1 = 0.4375$ as used in both models.

Lower values of δ are indicated by red colors and higher values by blue ones. The ν_1 used for both models is indicated by a white line. The complicated fine structure of the error function in these pictures demonstrates the complexity of the interrelation of *H/V* with its approximation by Love. However, it becomes obvious, that there is a greater "red island" (for $0.35 < \nu_1 < 0.45$ and $0.25 < d/\lambda_{\beta_1} < 0.4$) for model 1 with the higher beta contrast, where Love's approximation works quite well. It can be seen, that we are at the border of this island with our ν_1 value. This red island is much smaller for model 2. In addition it comes at no surprise, that in conformity with Fig. 5 we are always in the red range for sufficiently high d/λ_{β_1}.

5. Conclusions

It is well known that the dependence of the ellipticity of Rayleigh waves on frequency is very sensitive on the material properties of the propagation medium. Its study is important from a theoretical as well as from a practical perspective. The exact analytic formula derived here is an effective tool for doing this. Already the comparetively simple model of 1 layer over half-space, which nevertheless is important for practical applications such as site effect studies, yields a great variety of appearances. Not all of them are understood analytically yet. We find that Love's simple approximation can be profitably used for compressible media if the shear-wave contrast (i. e. the impedance contrast) between layer and half-space is high enough. For a completely unknown model, however, the exact formula has to be used.

The coincidence between the shear-wave resonance frequency and the frequency of the peak of the *H/V*- ratio, often used in practical applications, could be confirmed numerically for models with high impedance contrast. However, its analytical relation still remains to be proven with the exact formula being a natural starting-point.

References

[1] A. M. Correig and M. Urquizu, Some dynamical characteristics of microseism time-series, Geophys J. Int. 149, 589-598 (2002).

[2] E. J. Douze, Rayleigh waves in short-period seismic noise, Bull. Seism. Soc. Am. 54, 1197-1212 (1964).

[3] T. Ohmachi and T. Umezono, Rate of Rayleigh waves in microtremors. In: Irikura, K., Kudo, K., Okada, H. and Sasatani, T. (ed.), The Effects of Surface Geology on Seismic Motion, Balkema, Rotterdam (1998).

[4] P.–Y. Bard, ESG98 - Second international symposium on the effects of surface geology on seimic motion, Yokohama, Japan (1998).

[5] K. Kudo, Practical estimates of site response. Proceedings of the fifth international conference on seismic zonation, Nice (1995).

[6] M. Ohrnberger, F. Scherbaum, F. Krüger, R. Pelzing and S. K. Reamer, How good are shear wave velocity models in the Lower Rhine Embayment (NW-Germany) obtained from inversion of ambient vibrations?, Bolletino di Geofisica Teorica ed Applicata, submitted (2003).

[7] F. Scherbaum, K. G. Hinzen and M. Ohrnberger, Determination of shallow shear wave velocity profiles in the Cologne Germany area using ambient vibrations, Geophys. J. Int. 152, 597-612 (2003).

[8] L. M. Munirova and T. B. Yanovskaya, Spectral ratio of the horizontal and vertical Rayleigh wave components and its application to some problems of seismology, Izvestiya, Physics of the Solid Earth 37, 709-716 (2001), Translated from Fizika Zemli No. 9, 10-18 (2001).

[9] P. G. Malischewsky, F. Wuttke and A. Ziegert, The use of acoustic surface waves for non-destructive testing (in German), Schriftenreihe Werkstoffwissenschaften 17, 135-140, Verlag Dr. Köster, Berlin (2002).

[10] A. E. H. Love, Some Problems of Geodynamics, Cambridge University Press, Cambridge (1911), republished by Dover Publications, New York (1967).

[11] A. Ben-Menahem and S. J. Singh, Seismic Waves and Sources, Springer-Verlag, New York (1981).

[12] P. G. Malischewsky, Comment to "A new formula for the velocity of Rayleigh waves" by D. Nkemzi, Wave Motion 31, 93-96 (2000).

[13] H. F. Tiersten, Elastic surface waves guided by thin films, J. Appl. Phys. 40, 770-789 (1969).

[14] P. Bövik, A comparison between the Tiersten model and O(h) boundary conditions for elastic surface waves guided by thin layers, Transactions of the ASME 63, 162-167 (1996).
[15] P. Malischewsky, Surface Waves and Discontinuities, Elsevier, Amsterdam (1987).
[16] P. G. Malischewsky, Some special solutions of Rayleigh's equation and the reflections of body waves at a free surface, Geofísica Internacional 39, 155-160 (2000).
[17] H. M. Mooney and B. A. Bolt, Dispersive characteristics of the first three Rayleigh modes for a single surface layer, Bull. Seism. Soc. Am. 56, 43-67 (1966).
[18] Y. Nakamura, A method for dynamic characteristics estimation of subsurface sing microtremor on the ground surface, RTRI 30-1 (1998).
[19] M. Muciarelli, Reliability and applicability of Nakamura's technique using microtremors: An experimental approach, Journal of Earthquake Engineering, Vol. 2-4, Issue 1-14 (1998).
[20] P.-Y. Bard, Personal communication, Genua 2002.

Appendix

The auxiliary entity *y*

By introducing the abbreviations

$$
\begin{aligned}
g_1 &= p_2\, q_1\, k^2 (m_2\, m_4 + 2 f_1 m_1\, m_3)\ , \\
g_2 &= -2k^2\, p_1\, q_1\, (m_1\, m_2 - 2 m_3\, p_2\, q_2\, \delta\mu)\ , \\
g_3 &= k^2 (m_2\, m_5 - 2 f_1\, p_2\, q_2\, m_3{}^2) \\
g_4 &= 2k^2\, p_1\, q_1\, (m_1\, m_2 - 2 m_3\, p_2\, q_2\, \delta\mu)\ , \\
g_5 &= -2 p_1\, q_1\, k^2 (f_1\, m_1{}^2 + p_2\, q_2\, m_4\, \delta\mu), \\
g_6 &= 2k^2 p_1\, q_2\, (f_1\, m_1\, m_3 - m_5\, \delta\mu)
\end{aligned}
$$

with
$\delta\mu = \mu_1 - \mu_2\ ,\ \delta\rho = \rho_1 - \rho_2\ ,$

and

$$
\begin{aligned}
m_1 &= 2k^2\, \delta\mu + \omega^2\, \rho_2\ , \\
m_2 &= 2k^2\, \delta\mu - \omega^2\, \delta\rho\ , \\
m_3 &= 2k^2\, \delta\mu - \omega^2\, \rho_1\ , \\
m_4 &= -4k^2\, \delta\mu + 2\omega^2 (\rho_1 - \rho_2\, \beta_2{}^2 / \beta_1{}^2)\ , \\
m_5 &= 4k^4\, \delta\mu + \omega^4\, \delta\rho / \beta_1{}^2 + 2k^2\, \omega^2 [\rho_2\, (\beta_2{}^2/\beta_1{}^2 + 1) - 2\rho_1]
\end{aligned}
$$

the entity y is obtained in the form

$$
y = \frac{g_1 \cosh(d\, q_1) + g_2 \sinh(d\, p_1) + g_3 \sinh(d\, q_1)}{g_4 \cosh(d\, p_1) + g_5 \cosh(d\, q_1) + g_6 \sinh(d\, q_1)}\ .
$$