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A signal processing method for circular arrays

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The theory of spectral representations of stationary random processes can be a useful tool in signal processing. Using this theory, it is possible to derive simple methods of estimating the phase velocity of surface waves from observations made with a circular array. The methods are totally nondirectional, thus allowing the use of microseisms for exploration seismology. Furthermore, the methods can be extended to yield directional information about both correlated and uncorrelated signals.

INTRODUCTION

Measurement of the dispersion of surface waves is potentially a method of gaining information about the structure of the earth immediately below the point of measurement.

Toksöz (1964) investigated this by observing the dispersion of microseisms. However, when using most arrays, this problem is made difficult by the need to estimate simultaneously the direction of travel and the velocity of the microseismic waves, an especially difficult task when waves are traveling in several directions. Aki (1957, 1964) pointed out that the use of a special design of array, consisting of seismometers equally spaced on a circle and one seismometer at the center, could simplify processing to overcome these difficulties.

In this paper, we first provide a mathematical background for the problem by employing the theory of stationary stochastic processes and their spectral representations. With this background it is then shown that Aki's method, which involved averaging the observed correlations between the center seismometer and each seismometer on the circle, can be extended to the situation where waves arriving from several directions are correlated. Furthermore, a method is given for testing the assumptions which must be made in applying the technique. The final section gives some simple examples.

STATIONARY PROCESSES IN THE PLANE

We begin by considering stationary random processes $z(t, \xi)$ defined in time and the plane, where t is time and ξ is the location in the plane. These will have a spectral representation (see Yaglom, 1961),

$$z(t, \xi) = \int_{-\infty}^{\infty} \int_{R^2} \exp\{-i\omega t - i\kappa.\xi\} \cdot \cdot \zeta'(d\omega, d\kappa),$$

where ω is the frequency (in radians per unit time), κ the vector wavenumber (in radians per unit distance), and ζ' is a random spectral process with uncorrelated increments. While this representation was derived by Yaglom as a purely mathematical result, it can be given a physical interpretation. It states that any stationary process in time and space can be considered as a continuous sum of independent waves with different frequencies ω and wavenumbers κ. The random spectral measure ζ' gives the amplitude and phase of each of these waves. While all stationary processes can be represented in this way, we shall see below that it is not always the most useful representation.

It will be convenient to consider this representation in polar form. If $\xi = r(\cos\theta, \sin\theta)$ and $\kappa = k(\cos\phi, \sin\phi)$, then

$$z(t, r, \theta) = \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{-\pi}^{\pi} \exp\{-i\omega t - irk \cdot \cdot \cos(\theta - \phi)\}\, \zeta(d\omega, dk, d\phi).$$

We can make an assumption that at each frequency ω, the energy is concentrated at a single wavenumber, that is, the velocity is a single valued function of frequency. This corresponds to the spectral process

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ζ being concentrated on a curve $[\omega, k(\omega)]$, and we have

$$z(t, r, \theta) = \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \exp\{-i\omega t - irk(\omega) \cdot \cdot \cos(\theta - \phi)\} \zeta(d\omega, d\phi).$$

The assumption about the velocity is a key one. Later we give a method which tests whether it holds in practice.

A *frequency-direction spectral density* $f(\omega, \phi)$ can be defined by

$$f(\omega, \phi) d\omega d\phi = E|\zeta(d\omega, d\phi)|^2,$$

which gives the average energy at frequency ω arriving from direction ϕ.

Using this density, we may define a *spatial covariance function*

$$\Gamma(\omega, r, \theta) = \int_{-\pi}^{\pi} \exp\{irk(\omega)\cos(\theta - \phi)\} \cdot \cdot f(\omega, \phi) d\phi,$$

which measures the covariance at frequency ω between the signals observed at (say) the origin and (r, θ). It is a quantity easily measured in practice using either analog or digital correlation techniques. We shall be more interested in an *averaged covariance function,*

$$\begin{aligned}\gamma(\omega, r) &= (2\pi)^{-1} \int_{-\pi}^{\pi} \Gamma(\omega, r, \theta) d\theta \\ &= \int_{-\pi}^{\pi} (2\pi)^{-1} \int_{-\pi}^{\pi} \exp\{irk(\omega) \cdot \\ &\quad \cdot \cos(\theta - \phi)\} d\theta f(\omega, \phi) d\phi \\ &= J_0(rk(\omega)) \int_{-\pi}^{\pi} f(\omega, \phi) d\phi \\ &= J_0(rk(\omega)) f_0(\omega),\end{aligned}$$

where J_0 is the zero-order Bessel function of the first kind and

$$f_0(\omega) = \int_{-\pi}^{\pi} f(\omega, \phi) d\phi.$$

It can be seen that the averaged covariance function is not dependent upon the directional properties of the process. It should make no difference whether there is a single signal or many.

The spectrum of the process at a single point in space is $f_0(\omega)$ so that the averaged complex coherence would be $\rho(\omega, r) = J_0[rk(\omega)]$. Aki (1957) suggested using this fact by having an array of seismometers equally spaced on a circle of radius r and having an extra seismometer at the center. At each frequency of interest, the coherence between each of the seismometers on the circle and the one in the middle was averaged to obtain an estimate $\hat{\rho}(\omega, r)$ of the averaged complex coherence. Then an estimate $\hat{k}(\omega)$ can be formed for the wavenumber by letting

$$J_0[r\hat{k}(\omega)] = \text{Real}(\hat{\rho}(\omega, r)].$$

In practice, we would want to be able to observe as wide a frequency range as possible while keeping the radius of the array fixed. The function $J_0[rk(\omega)]$ is one-to-one for $rk(\omega)$ over the range 0 to $\simeq 3.8$. But in practice, not all of this range can be used since at each end any errors are greatly magnified because

$$\text{Variance}[\hat{k}(\omega)] \approx \text{Variance}[\hat{\rho}(\omega, r)] \cdot \cdot \{rJ_1[rk(\omega)]\}^{-2}.$$

This usually means that $rk(\omega)$ must lie in the range .4 to 3.2, or alternatively the wavelengths must lie between $2r$ and $15r$.

Aki suggested that $\rho(\omega, r)$ could be estimated by considering the coherence between the center seismometer and the average of the outputs of the seismometers on the circle. We now show that this is not correct, but it does lead to a method of testing for nonpropagating noise or multiple velocities.

We shall define the center signal by

$$x_0(t) = z(t, 0, 0),$$

and the circle signal by

$$y_0(t) = (2\pi)^{-1} \int_{-\pi}^{\pi} z(t, r, \theta) d\theta.$$

Expressed in terms of the spectral process ζ, they are

$$x_0(t) = \int_{-\infty}^{\infty} \exp\{-i\omega t\} \int_{-\pi}^{\pi} \zeta(d\omega, d\phi),$$

and

$$\begin{aligned}y_0(t) &= \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} (2\pi)^{-1} \int_{-\pi}^{\infty} \exp\{-i\omega t \\ &\quad - irk(\omega) \cos(\theta - \phi)\} d\theta \, \zeta(d\omega, d\phi) \\ &= \int_{-\infty}^{\infty} \exp\{-i\omega t\} J_0[rk(\omega)] \cdot \\ &\quad \cdot \int_{-\pi}^{\pi} \zeta(d\omega, d\phi).\end{aligned}$$

If we let

$$\int_{-\pi}^{\pi} \zeta(\omega, d\phi) = d\zeta_0(\omega),$$

then these simplify to

$$x_0(t) = \int_{-\infty}^{\infty} \exp\{-i\omega t\} d\zeta_0(\omega),$$

and

$$y_0(t) = \int_{-\infty}^{\infty} \exp\{-i\omega t\} J_0[rk(\omega)]\, d\zeta_0(\omega).$$

This implies that the circle signal $y_0(t)$ can be regarded as a linearly filtered form of the center signal $x_0(t)$, with gain function $J_0[rk(\omega)]$ (see, for example, Koopmans, 1974, p. 86). Except when $J_0[rk(\omega)] = 0$, the complex coherence between the two signals will be 1 or -1 depending on the sign of $J_0[rk(\omega)]$. Furthermore, the introduction of either noise or another significant velocity will reduce this coherence to a value less than one.

There are two practical outcomes of this. First, by estimating the coherence between the center signal and the circle signal (in practice obtained by averaging the outputs of the seismometers on the circle), a check can be made on the assumptions necessary to apply these techniques. Second, if the gain function is estimated [considering $x_0(t)$ as the input to the filter and $y_0(t)$ as the output], it could be used as an alternative estimate of $J_0[rk(\omega)]$, and hence $k(\omega)$ may be estimated. Any standard procedure for estimating the gain may be used (see, for example, Brillinger, 1975, p. 302). However, this method of estimating $k(\omega)$ has one disadvantage over the method that Aki proposed—the seismometers would have to be precisely calibrated. If the seismometer gains vary, it is even possible to have an estimated gain function which is greater than unity at some frequencies. Obviously, no sense can be made of this. However, as we shall see in the next section, a slight extension of the technique makes this disadvantage worth overcoming.

FOURIER-BESSEL ANALYSIS USING A CIRCULAR ARRAY

The terms $f_0(\omega)$ and $\zeta_0(\omega)$ may be regarded as the zero-order terms in the Fourier series of $f(\omega, \phi)$ and $\zeta(\omega, \phi)$, respectively. The higher order terms will be defined by

$$f_m(\omega) = \int_{-\pi}^{\pi} \exp\{-im\phi\} f(\omega, \phi)\, d\phi,$$

and

$$\zeta_m(\omega) = \int_{-\pi}^{\pi} \exp\{-im\phi\}\, \zeta(\omega, d\phi).$$

Then the inverse relations are

$$f(\omega, \phi) = (2\pi)^{-1} \sum_{m=-\infty}^{\infty} \exp\{im\phi\} f_m(\omega),$$

and

$$\zeta(\omega, d\phi) = (2\pi)^{-1} \sum_{m=-\infty}^{\infty} \exp\{im\phi\}\, \zeta_m(\omega)\, d\phi.$$

Furthermore, the original process $z(t, r, \theta)$ can be expressed in terms of

$$\begin{aligned} z(t, r, \theta) &= \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \exp\{-i\omega t\} \cdot \\ &\quad \cdot (2\pi)^{-1} \int_{-\pi}^{\pi} \exp\{-irk(\omega) \cdot \\ &\quad \cdot \cos(\theta - \phi) + im\phi\}\, d\phi\, d\zeta_m(\omega) \\ &= \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \exp\{-i\omega t + im\theta\} \cdot \\ &\quad \cdot J_m[rk(\omega)]\, d\zeta_m(\omega). \end{aligned}$$

Now we define what we shall call the mth order circle process:

$$\begin{aligned} y_m(t) &= (2\pi)^{-1} \int_{-\pi}^{\infty} \exp\{-im\theta\}\, z(t, r, \theta)\, d\theta \\ &= \int_{-\infty}^{\infty} \exp\{-i\omega t\}\, J_m(rk(\omega))\, d\zeta_m(\omega). \end{aligned}$$

Hence, the circle signals provide a method of isolating the Fourier components of the spectral process.

Before examining this further, we shall introduce a slight extension to cover some cases where energy arriving from different directions is correlated. Two obvious cases are that of a signal and its echo and that of a scattering medium. In both these cases, it is reasonable to assume that the observed signals are the result of unobserved uncorrelated signals undergoing some form of scattering which alters the directional distribution of energy. The only model for which the mathematics is workable is that where this scattering is linear and isotropic (that is, the same for signals arriving from all directions). With such a model, the spectral process of the observed process can be represented as a convolution of the original spectral process with some suitable function.

We define a *scattering function* $W(\omega, \psi)$, and a new spectral process ζ^0, by

$$\zeta^0(\omega, \phi) = \int_{-\pi}^{\pi} W(\omega, \phi - \psi)\, \zeta(\omega, d\psi).$$

This equation implies that $d\zeta^0(\omega, \phi)$, the observed energy at frequency ω from direction ϕ, is generated by summing the contributions of an original unscattered process ζ. The sum is weighted by the function W. Note that in general ζ^0 does not have orthogonal or uncorrelated increments in ϕ. Hence, our representation is no longer the standard one of Yaglom.

Usually $W(\omega, \psi)$, considered as a function of ψ, would have a single main peak at zero, implying that most of the energy is only slightly scattered. It could be complex valued if the scattering involves time delays. We shall assume that $W(\omega, \psi)$ has a Fourier expansion with respect to ψ, that is

$$W(\omega, \psi) = (2\pi)^{-1} \sum_{m=-\infty}^{\infty} \exp\{im\psi\} C_m(\omega).$$

Since the unscattered process is not observed, we may assume that $C_0(\omega) = 1$. If the scattering is significant and $W(\omega, \psi)$ is thus a smooth function in ψ, we would expect the higher order coefficients $C_m(\omega)$ to be relatively small.

Since $\zeta^0(\omega, \phi)$ is defined by a convolution, we have

$$\zeta^0(\omega, d\phi) = (2\pi)^{-1} \sum_{m=-\infty}^{\infty} \exp\{im\phi\} \cdot \cdot C_m(\omega)\, \zeta_m(\omega)\, d\phi,$$

$$z(t, r, \theta) = \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \exp\{-i\omega t + im\theta\} \cdot \cdot J_m[rk(\omega)]\, C_m(\omega)\, d\zeta_m(\omega),$$

and

$$y_m(t) = \int_{-\infty}^{\infty} \exp\{-i\omega t\} \cdot \cdot J_m[rk(\omega)]\, C_m(\omega)\, d\zeta_m(\omega).$$

Now we can proceed to examine the properties of these circle signals. First, we note that

$$\begin{aligned} E(\zeta_\ell(d\omega)\, \overline{\zeta_m(d\omega)}) &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cdot \\ &\quad \cdot \exp\{-i\ell\phi + im\phi'\} \cdot \\ &\quad \cdot E(\zeta d\omega, d\phi)\, \overline{\zeta(d\omega, d\phi')}) \\ &= \int_{-\pi}^{\pi} \exp\{-i(\ell - m)\} \cdot \\ &\quad \cdot f(\omega, \phi)\, d\phi\, d\omega \\ &= f_{\ell-m}(\omega)\, d\omega. \end{aligned}$$

Since we can assume that $C_0(\omega) = 1$, the center signal will be defined as before. Then the covariance relations between the circle signals and the center signals will be

$$\begin{aligned} E(y_\ell(s)\, \overline{y_m(s+t)}] &= \int_{-\infty}^{\infty} \exp\{i\omega t\} \cdot \\ &\quad \cdot J_\ell[rk(\omega)]\, J_m[rk(\omega)] \cdot \\ &\quad \cdot C_\ell(\omega)\, \overline{C_m(\omega)}\, f_{\ell-m}(\omega)\, d\omega, \\ E[x_0(s)\, \overline{y_m(s+t)}] &= \int_{-\infty}^{\infty} \exp\{i\omega t\} \cdot \\ &\quad \cdot J_m[rk(\omega)]\, \overline{C_m(\omega)} \cdot \\ &\quad \cdot f_{-m}(\omega)\, d\omega, \end{aligned}$$

and

$$E[x_0(s)\, \overline{x_0(s+t)}] = \int_{-\infty}^{\infty} \exp\{i\omega t\}\, f_0(\omega)\, d\omega.$$

As in the previous section, it seems most useful to compare the center signal with each of the circle signals. The bivariate signal $[x_0(t), y_m(t)]$ has a spectral density matrix $G(\omega)$ given by

$$G(\omega) = \begin{bmatrix} f_0(\omega) & J_m[rk(\omega)]\overline{C_m(\omega)} f_{-m}(\omega) \\ J_m[rk(\omega)] C_m(\omega) f_m(\omega) & J_m[rk(\omega)]^2 |C_m(\omega)|^2 f_0(\omega) \end{bmatrix}$$

Note that in the case $m = 0$, $G(\omega)$ is not affected by the presence of scattering [since $C_0(\omega) = 1$] and hence the results of the previous section are independent of this type of correlation in the observed signals.

PRACTICAL CONSIDERATIONS

Consider a practical array with p seismometers equally spaced on a circle. Obviously, the integral definition of the circle signals cannot be used and instead a finite sum must be formed. If we let the polar coordinates of the seismometers be $(r, 2\pi ap^{-1})$, $a = 1, 2, \ldots, p$, then the mth order circle process will be

$$\tilde{y}_m(t) = p^{-1} \sum_{a=1}^{p} z(t, r, 2\pi ap^{-1}) \cdot \cdot \exp\{-i2\pi map^{-1}\}.$$

This need only be calculated for $m = -\tfrac{1}{2}p, \ldots, \tfrac{1}{2}p$ since a form of directional aliasing will occur, and

$$y_m(t) = \sum_k y_{m+pk}(t)$$

$$= \int_{-\infty}^{\infty} \sum_k \exp\{-i\omega t\} J_{m+pk}[rk(\omega)] \cdot$$
$$\cdot C_{m+pk}(\omega)\, d\zeta_{m+pk}(\omega).$$

However, $J_n(x)$ tends to zero very rapidly as n tends to infinity so that this aliasing will not be significant for reasonable values of p except at those frequencies where $J_m[rk(\omega)]$ is close to zero. At those frequencies, the higher order aliased terms will dominate and confuse the results.

We now consider the bivariate signal $[x_0(t), y_m(t)]$. This may either be analyzed in real time using narrow-band filters or alternatively be recorded and processed using the fast Fourier transform. Either way, an estimate of the spectral density matrix $G(\omega)$ can be obtained (see, for example, Koopmans, 1974). Using this estimate of $G(\omega)$ we may define the following functions:

the mth order *circle coherence*,

$$\sigma_m(\omega) = |G_{12}(\omega)| [G_{11}(\omega) G_{22}(\omega)]^{-1/2},$$

the mth order *circle phase*,

$$\theta_m(\omega) = \text{phase}\, G_{12}(\omega),$$

and the mth order *scatter function*,

$$S_m(\omega) = G_{22}(\omega)/G_{11}(\omega).$$

Observe that asymptotically (as the spectral density estimates tend to the true values) $\sigma_m(\omega)$ approaches $|f_m(\omega)|/f_0(\omega)$, θ_m approaches phase $[\bar{C}_m(\omega)]$ + phase $[f_{-m}(\omega)]$, and $S_m(\omega)$ approaches $J_m[rk(\omega)]^2 |C_m(\omega)|^2$. Unfortunately, it is not possible to estimate the phases of $f_m(\omega)$ and $C_m(\omega)$ separately since the unscattered process is not observed. The following examples will demonstrate the usefulness of these functions.

Example 1

The first example worth considering is that of a single signal arriving from direction ϕ_0 observed without scattering. Then

$$f(w, \phi) = f(\omega)\, \delta(\phi - \phi_0),$$

and

$$W(\omega, \psi) = \delta(\psi),$$

where δ is the delta function. This gives

$$f_0(\omega) = f(\omega),$$
$$f_m(\omega) = f(\omega) \int \exp\{-im\phi\}\, \delta(\phi - \phi_0)\, d\psi = f(\omega) \exp\{-im\phi_0\},$$

and

$$C_m(\omega) = 1.$$

The asymptotic values of the functions σ_m, θ_m and S_m are given by

$$\sigma_m(\omega) = 1,$$
$$\theta_m(\omega) = -m\phi_0,$$

and

$$S_m(\omega) = J_m(rk(\omega))^2.$$

In practice these functions can be calculated, and it would be easy to decide if they are of the above form. If so, the direction of the signal can be easily estimated from $\theta_1(\omega)$. It is interesting to compare this method with virtually all other methods of finding directions of signals. Most of them involve finding the maximum of some function (an estimated wavenumber spectrum, for example) while the above method only requires evaluation of an explicit function.

Example 2

If two signals are present, the problem is much more difficult. We would have

$$f(\omega, \phi) = f(\omega)\, \delta(\phi - \phi_1) + g(\omega)\, \delta(\phi - \phi_2),$$
$$f_0(\omega) = f(\omega) + g(\omega),$$

and

$$f_m(\omega) = f(\omega) \exp\{-im\phi_1\} + g(\omega) \exp\{-im\phi_2\}.$$

The functions σ_m, θ_m and S_m will now be hard to interpret. However, it is obvious that $\sigma_m(\omega) < 1$, $m \neq 0$ at most frequencies, and in practice this could be used as an indication of the presence of more than one signal.

Example 3

If the scattering function is simply assumed to be real and symmetric about zero, then the situation is only slightly modified. In this case the coefficients C_m are real and $C_m(\omega) = C_{-m}(\omega)$, and the functions $\sigma_m(\omega)$ and $\theta_m(\omega)$ will be unchanged. The presence of scattering will be demonstrated by the change in $S_m(\omega)$. If the function has a smooth broad peak at zero, it will usually be found that $|C_m(\omega)| < 1$ for $m \neq 0$, and consequently $S_m(\omega)$ will be small for $m \neq 0$. In general, small observed values of $S_m(\omega)$ will imply a high degree of scattering.

CONCLUSIONS

This paper developed first a physical interpretation of the spectral representation of stationary process to provide a context in which earlier results of Aki could be analyzed. It was then possible to obtain an alternative method of estimating wavenumber functions and a test for the validity of the methods.

The second part extended the methods to obtain directional information from the data. This was done by deriving explicitly defined functions which are often readily interpreted. This is in contrast to almost all other methods of array processing which require nonlinear maximization of functions and which do not identify correlations between waves from different directions. The appropriate context for these methods appears to be in obtaining a quick analysis of data, and as a starting point for more refined methods.

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