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22. *Space and Time Spectra of Stationary Stochastic Waves, with Special Reference to Microtremors.*

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(Read May 28, 1957.—Received June 30, 1957.)

Introduction

Since the days of Wiechert and Galitzin, seismograms have chiefly been investigated from the view point that they consist of successive distinguishable phases, and the travel time curves for various phases have been essential clues in revealing the structure and state of the matter within the earth. This idea of "phase" is indeed very natural and appropriate so far as the duration of shocks at their origin is negligibly short as compared with the characteristic time of the structure, such as a crustal layer, through which the seismic waves are propagated. Here the characteristic time of a layer may be represented by the ratio of its thickness to the velocity of seismic wave propagation in the layer.

There are, however, many cases in which the above assumption of short duration of shocks does not hold. An example of such cases is the propagation of seismic waves through a complicated crust. What can be clearly identified on the records of seismic waves due to near earthquakes such as those frequently observed in Japan is the initial motion of P waves and at best that of S waves. The main remaining part of such a seismogram has not been paid due attentions, if not neglected, any information from this source regarding the nature of the medium of propagation having been scarcely expected.

Other examples are the waves due to causes other than earthquakes, such as microseismic waves closely connected with meteorological disturbances, volcanic tremors, microtremors generated by traffic, and other tremors of artificial origin. It is hardly possible to deal with those waves from the standpoint of phases and to deduce from them any useful travel time curves.

The object of the present paper is to develop a method for dealing with those complicated waves in order that the nature of the waves as well as the nature of the medium of propagation may be revealed. Since the method is based on a statistical investigation of waves in time and in space, we need to assume that our waves are stationary in both. This assumption represents quite an opposite extremity as compared with that underlying the phase method, and is certainly an appropriate one for studying those complicated waves mentioned above.

It is true that many studies on such waves have been made by various authors from the statistical point of view. But so far as the writer is aware, those studies have been made for rather limited purposes. For example, the study of spectral distribution of seismic waves has aimed at either getting useful information for earthquake damage prevention or investigating the dependence of the spectrum on the epicentral distance, the earthquake magnitude and the nature of wave paths and so on. Similar studies have also been made about volcanic tremors as well as microtremors due to traffic origin, and the spectrum of microseismic waves has been studied in reference to that of sea waves which are believed to cause them. Also the object of the use of filters in explosion seismology has been to secure a clearer identification of phases on a seismogram.

Those studies have been primarily concerned with the spectrum of waves in time, while the spectrum in space has not yet attracted due attentions. The recent study by K. Akamatsu[1] (1956) of the autocorrelation of microtremor waves in space is among the few made on the latter subject. She has made clear the spatial character of vibration of the ground. The process for obtaining the spatial autocorrelation coefficient, however, consists of troublesome steps such as simultanous recordings of vibrations at several points, readings of the recorded amplitudes, and computations of the autocorrelation coefficient among the waves to be studied. In order to secure rapidness and efficiency of measurments in the study of this kind, K. Aki[2] (1956) built a simple automatic computer by which the computation of spatial autocorrelation coefficients can be made without following individual steps stated above.

So far as the writer knows, the study to be reported here is the first, specifically designed to elucidate the relation between the spectrum of waves in space and that in time with reference to the nature of medium of propagation. By means of the method presented in this paper, the direction distribution of propagation as well as the mode of

1) K. AKAMATU, *Zisin*. [ii], **9** (1956), 22.
2) K. AKI, *ibid*. **9** (1956), 40.

polarization of complicated waves can be learned as will be seen in later chapters. Also we can obtain the dispersion curves for those waves which are useful for deducing the structure of the medium. In addition to this, if the waves observed consist of partial waves having different velocities, the power of each waves can be found.

In Chapter 1 will be given, some results of theoretical considerations of stochastic waves which are stationary both in time and space, and it will be shown what should be measured in order to find the dispersion curves, the mode of polarization, etc. of the waves. The instruments designed specifically for this purpose will be described in Chapter 2. They consists of filters of phase shift type and an automatic computer of the correlation coefficient.

Chapter 3 will be devoted to describing the results of application of the present method to the study of microtremors due to traffic observed at Hongo, Tokyo. The results obtained are as follows; 1) those waves are propagating in every direction with almost uniform power; 2) the horizontal component of vibration is strongly polarized in the direction perpendicular to the direction of propagation showing that they are of Love type; 3) the dispersion curves have been deduced, and the velocities of S waves at various depths calculated.

Chapter 1. Theory of stationary stochastic waves

The most fundamental material in the study of wave from the standpoint of phases is certainly the travel time curve which indicates the relation between the travel time and epicentral distance. It may be expected that the corresponding fundamental material in the spectral studies of waves will be a certain relation between the spectrum of the waves in space and that in time. At first we shall look for this relation in the most simplified case of one dimensional waves, and at the same time shall attempt to show the characteristics of stochastic waves which are stationary in time and space.

1. One dimensional stationary waves having one single velocity

With the assumption that our waves travel with a single and definite velocity c independent of the frequency of vibration, our waves $u(x, t)$ can be expressed for the region $x=0 \sim X$ formally

$c\rho_n t = \frac{2\pi n c t}{X}$ $\omega_n = \frac{2\pi n c}{X}$ $\rightarrow i\rho_n x = i\frac{\omega_n x}{c}$

$$u(x,\ t)=\sum A_n \exp(i\rho_n x)\cos c\rho_n t$$
$$+\sum \frac{B_n}{c\rho_n}\exp(i\rho_n x)\sin c\rho_n t \tag{1}$$

where

$$\rho_n = 2\pi\frac{n}{X} \quad (n=0,\ \pm1,\ \pm2,\cdots)$$

This is the solution of the one dimensional wave equation under the initial conditions that

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$$\left.\begin{aligned} u(x,\ 0)&=\sum A_n \exp(i\rho_n x)\\ \dot{u}(x,\ 0)&=\sum B_n \exp(i\rho_n x)\ . \end{aligned}\right\} \tag{2}$$

Since $u(x,\ 0)$ and $\dot{u}(x,\ 0)$ are both real, A_n and B_n must be the conjugate complex numbers of A_{-n} and B_{-n} respectively.

Now let us find the condition under which the waves formally given by Eq. (1) are stationary both in time and in space. At first, we notice the initial state of our waves as given by Eq. (2). Here $u(x,\ 0)$ and $\dot{u}(x,\ 0)$ should be treated as stochastic variables with a parameter x.

The Fourier coefficient A_n of a general stochastic process which is stationary with respect to a single parameter x for the region $x=0 \sim X$ is known to be written in terms of the corresponding Fourier coefficient E_n of the so called "thermal or white noise" as follows;

$$A_n = E_n^{(A)}\cdot G^{(A)}(\rho_n)\ , \tag{3}$$

where $G^{(A)}(\rho_n)$ is not a stochastic variable. From the purely random character of "white noise", it follows that

$$\left.\begin{aligned} &\overline{E_n\cdot E_m}=0\ , \quad n+m \neq 0\ ,\\ &\overline{E_n\cdot E_{-n}}=\overline{|E_n|^2}=\frac{\Delta\rho_n}{2\pi}=\frac{1}{X}\ , \end{aligned}\right\} \tag{4}$$

where the bars represent the operation of average.

Using these formulas, we have

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$$\left.\begin{aligned} &\overline{A_n A_m}=0\,, \quad n+m \neq 0\,, \\ &\overline{|A_n|^2}=|G^{(A)}(\rho_n)|^2 \frac{\Delta\rho_n}{2\pi}\,, \end{aligned}\right\}$$

and in a similar way, (5)

$$\left.\begin{aligned} &\overline{B_n B_m}=0\,, \quad n+m \neq 0 \\ &\overline{|B_n|^2}=|G^{(B)}(\rho_n)|^2 \frac{\Delta\rho_n}{2\pi}\,. \end{aligned}\right\}$$

These are the statistical relations existing among the Fourier coefficients A_n's and B_n's. Moreover, if the initial distributions of displacement and that of velocity are independent of each other, we have

$$\overline{A_n B_m}=0\,, \tag{6}$$

for all n and m. From Eqs. (2) and (5), we see that $|G^{(A)}(\rho_n)|^2$ and $|G^{B}(\rho_n)|^2$ represent the spatial spectrum density of the initial displacement and that of the initial velocity respectively.

Defining the spatial autocorrelation function $\phi(\xi, t)$ of our waves for a given time t as

$$\phi(\xi, t)=\overline{u(x, t)u(x+\xi, t)} \tag{7}$$

and using Eqs. (5) and (6), we obtain

$$\begin{aligned} \phi(\xi, t) &= \sum \frac{\Delta\rho_n}{2\pi}\left\{|G^A(\rho_n)|^2 \cos^2 c\rho_n t + \frac{|G^B(\rho_n)|^2}{c^2\rho_n^2} \sin^2 c\rho_n t\right\} \exp(i\rho_n\xi) \\ &= \frac{1}{2\pi}\int \{|G^A(\rho_n)|^2 \cos^2 c\rho_n t + \frac{|G^B(\rho_n)|^2}{c^2\rho_n^2} \sin^2 c\rho_n t\} \exp(i\rho_n\xi)d\rho \end{aligned} \tag{8}$$

From Eq. (8), we see that if

$$\left.\begin{aligned} &|G^A(\rho_n)|^2=\frac{|G^B(\rho_n)|^2}{c^2\rho_n^2} \\ \text{or}\quad & \\ &\omega_n|A_n|=|B_n| \end{aligned}\right\} \tag{9}$$

where ω_n is the circular frequency and $\omega_n=c\rho_n$, $\phi(\xi, t)$ becomes independent of time. Thus we can reasonably take Eq. (9) as the condition for stationary stochastic waves. Eq. (9) can be considered as representing the law of equipartition of energy in the case of stochastic waves.

Introducing this condition into Eq. (8) and dropping the suffix A we get

$$\boxed{\phi(\xi,\ t) = \phi(\xi) = \frac{1}{2\pi}\int |G(p)|^2 \exp(ip\xi)\,dp} \tag{10}$$

We shall now proceed to investigate the relation between the spectrum in space and that in time. For this we define the spectrum density in time as

$$\Phi(\omega_n) = \frac{1}{4}\cdot\frac{\{U_c(\omega_n)\}^2+\{U_s(\omega_n)\}^2}{\Delta\omega_n/2\pi} \tag{11}$$

where $U_c(\omega_n)$ is the Fourier cosine coefficient of $u(x,\ t)$ with respect to t and for a given x, while $U_s(\omega_n)$ is the corresponding sine coefficient. It can readily be seen from Eq. (1) that

$$\left.\begin{aligned} U_c(\omega_n) &= A_n \exp\left(i\frac{\omega_n}{c}x\right) + A_{-n}\exp\left(-i\frac{\omega_n}{c}x\right) \\ U_s(\omega_n) &= \frac{B_n}{\omega_n}\exp\left(i\frac{\omega_n}{c}x\right) + \frac{B_{-n}}{\omega_n}\exp\left(-i\frac{\omega_n}{c}x\right) \end{aligned}\right\} \tag{12}$$

and

$$\Delta W_n = C\Delta\rho_n = \frac{2\pi c}{X}$$

$$\Delta\omega_n = c\Delta p_n = \frac{2\pi c}{X}$$

Inserting Eq. (12) into Eq. (11), we get

$$\Phi(\omega_n) = \frac{\left[A_n\exp\left(i\frac{\omega_n}{c}x\right)+A_{-n}\exp\left(-i\frac{\omega_n}{c}x\right)\right]^2+\left[\frac{B_n}{\omega_n}\exp\left(i\frac{\omega_n}{c}x\right)+\frac{B_{-n}}{\omega_n}\exp\left(-i\frac{\omega_n}{c}x\right)\right]^2}{4\Delta\omega_n\, 2\pi}$$

By the use of Eq. (5) this may be written as

$$\Phi(\omega_n) = \frac{2A_nA_{-n}+2\overline{B_nB_{-n}}/\omega_n^2}{4\Delta\omega_n/2\pi}$$

Finally, inserting Eq. (9) into this, we obtain the following equation,

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e following equation,

$$\boxed{\Phi(\omega)=\frac{|G(\rho)|^2\Delta\rho/2\pi}{\Delta\omega/2\pi}=\frac{|G(\omega/c)|^2}{c}} \tag{14}$$

where the suffix n is dropped. This is the relation which connects the spectrum density in space and that in time in the case of one dimensional waves.

As will be shown later, as compared with Eq. (14), the following equation which relates the spatial autocorrelation function $\phi(\xi)$ with the spectrum density $\Phi(\omega)$ in time is more convenient for the purpose of the present study,

$$\phi(\xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\omega)\exp\left(i\frac{\omega}{c}\xi\right)d\omega$$

or (15)

$$\phi(\xi)=\frac{1}{\pi}\int_{0}^{\infty}\Phi(\omega)\cos\left(\frac{\omega}{c}\xi\right)d\omega$$

which can readily be obtained by Eqs. (10) and (14).

2. Dispersive waves

We shall now proceed to the case of dispersive waves, and show that Eq. (15) obtained above holds also in this case without any modification except the substitution of the function $c(\omega)$ of frequency ω for the constant velocity c. For this, we notice that if we take $\Delta\rho$ as constant for all n, the interval $\Delta\omega_n$ between consecutive ω_n is no longer constant in the dispersive case and varies with n. Then we may write

$$\Delta\omega_n=\left(\frac{d\omega}{d\rho}\right)_n\Delta\rho_n\,. \tag{16}$$

The equation corresponding to Eq. (14) is now written as

$$\Phi(\omega)=\frac{|G(\omega/c)|^2}{d\omega/d\rho}\,. \tag{17}$$

Introducing of this into Eq. (10) yields the final formula,

$$\phi(\xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}|G(\omega,c)|^2 \exp\left(\frac{i\omega}{c(\omega)}\xi\right)\frac{d\rho}{d\omega}\,d\omega$$
$$=\frac{1}{\pi}\int_0^{\infty}\Phi(\omega)\cos\left(\frac{\omega}{c(\omega)}\xi\right)d\omega \tag{18}$$

3. Spatial autocorrelations of filtered waves (1)

In this section the most essential part of our method will be illustrated in the case of one dimensional waves. Corresponding to the separation of a seismogram into successive particular phases in a study of "phases", the vibration of a seismograph is resolved into simple harmonic oscillations; in other words, a Fourier analysis is applied to the vibration in this method. For this purpose, we use electronic resonators to which we shall refer in Chapter 2. If the filtration by a resonator having frequency ω_0 is sufficiently sharp to allow us to assume the spectrum density of the filtered vibration to be

$$\Phi(\omega)=P(\omega_0)\delta(\omega-\omega_0)\,, \quad \omega>0 \tag{19}$$

where $\delta(\omega)$ is the Dirac δ-function, then the corresponding spatial autocorrelation function (18) is written as

$$\phi(\xi,\omega_0)=P(\omega_0)\cos\left(\frac{\omega_0}{c(\omega_0)}\xi\right). \tag{20}$$

Defining the autocorrelation coefficient as

$$\rho(\xi,\omega_0)=\frac{\phi(\xi,\omega_0)}{\phi(0,\omega_0)}\,,$$

we may write it as

$$\rho(\xi,\omega_0)=\cos\left(\frac{\omega_0}{c(\omega_0)}\xi\right). \tag{21}$$

The above formula shows that the dispersion curve i.e. the curve of velocity $c(\omega_0)$ as a function of frequency ω_0 can be obtained directly from the measurement of $\rho(\xi,\omega_0)$. The measurement of this quantity $\rho(\xi,\omega_0)$ for various frequencies ω_0 and for a fixed distance ξ is therefore the most fundamental in our method. But this is allowed only in the case in which the waves concerned have a single velocity corresponding

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Next, we shall consider the wave which is composed of partial waves having different velocities. In this case if we are allowed to assume that the component waves are statistically independent of one another, we can separate the composite waves into the components by measuring $\rho(\xi, \omega_0)$. Now we write the quantities related to the n'th component wave by attaching the suffix n: for instance, the displacement of the n'th component as $u_n(x, t)$ and the corresponding velocity as $c_n(\omega)$. Then from the assumption of independence among the components, it follows that

$$u(x, t)=\sum_n u_n(x, t)\,,$$

$$\phi(\xi)=\sum_n \phi_n(\xi)=\sum \frac{1}{2\pi}\int \Phi_n(\omega)\exp\left(i\frac{\omega}{c_n(\omega)}\xi\right)d\omega\,,$$

$$\phi(\xi, \omega_0)=\sum_n P_n(\omega_0)\cos\left(\frac{\omega_0}{c_n(\omega_0)}\xi\right),$$

$$\rho(\xi, \omega_0)=\sum_n \frac{P_n(\omega_0)}{P(\omega_0)}\cos\left(\frac{\omega_0}{c_n(\omega_0)}\xi\right). \tag{22}$$

The last equation shows that the number N of components is, finite, we can in principle obtain both the percentage of power of the n'th component and the corresponding velocity from the value of $\rho(\xi, \omega_0)$ for a given ω_0 and for $(2N-1)$ different ξ's.

Finally, there may be cases in which the wave can be assumed as composed of component waves having continuously distributed velocity. In this case, introducing the velocity distribution function defined as

$$p(\omega, c)=P_n(\omega)/\Delta c_n$$

and replacing the summation by the integration in Eq. (22), we have

$$\rho(\xi, \omega)=\frac{1}{P(\omega)}\int_0^\infty p(\omega, c)\cos\left(\frac{\omega}{c(\omega)}\xi\right)dc\,. \tag{23}$$

From this and using the Fourier transformation, we obtain

$$\frac{c^2 p(\omega, c)}{P(\omega)}=\frac{2}{\pi}\int_0^\infty \rho(\xi, \omega)\cos\left(\frac{\omega}{c(\omega)}\xi\right)d\xi\,. \tag{24}$$

Therefore, if we find the values of $\rho(\xi, \omega)$ for $\xi=0\sim\infty$, we can calculate the percentage of power of component wave having the assigned velocity and frequency contained in the wave in question.

4. Two dimensional waves of a single velocity

Now let us investigate the case of two dimensional waves, in which the reasonning is quite analogous to that given above for the case of one dimensional waves, though there appear additional terms such as the direction of propagation and polarization of vibrations. At first we shall deal with those waves which are neither dispersive nor polarized.

Assuming that our waves travel with a single velocity c, we write them in the form

$$u(x,y,t)=\sum\sum A_{nm}\exp(i\rho_n x\cos\theta_m+i\rho_n y\sin\theta_m)\cos(c\rho_n t)$$
$$+\sum\sum\frac{B_{nm}}{c\rho_n}\exp(i\rho_n x\cos\theta_m+i\rho_n y\sin\theta_m)\sin(c\rho_n t)\,. \quad (25)$$

This is the solution of a two dimensional wave equation under the initial conditions that

$$\left.\begin{aligned} u(x,y,0)&=\sum\sum A_{nm}\exp(i\rho_n x\cos\theta_m+i\rho_n y\sin\theta_m)\\ \dot{u}(x,y,0)&=\sum\sum B_{nm}\exp(i\rho_n x\cos\theta_m+i\rho_n y\sin\theta_m)\end{aligned}\right\}. \quad (26)$$

Since $u(x,y,0)$ and $\dot{u}(x,y,0)$ are both real, $A_{n.m}$ and $B_{n.m}$ are the conjugate complex number of $A_{n.m+(\pi)}$ and $B_{n.m+(\pi)}$ respectively, in which (π) is a suffix defined by the relation, $\theta_{(\pi)}=\pi$.

Analogous to Eq. (5), the mean value of the absolute square of $A_{n.m}$ and that of $B_{n.m}$ are written as

$$\left.\begin{aligned} |A_{nm}|^2&=|G^A(\rho_n,\theta_m)|^2\frac{\rho_n\Delta\theta_m\Delta\rho_n}{(2\pi)^2}\\ |B_{nm}|^2&=|G^B(\rho_n,\theta_m)|^2\frac{\rho_n\Delta\theta_m\Delta\rho_n}{(2\pi)^2}\end{aligned}\right\}. \quad (27)$$

The spectrum density $|G(\rho,\theta)|^2$ in the above equation represents the amount of power carried in the waves at the initial state per unit area of the phase space which is formed by two dimensional wave numbers $\lambda=\rho\cos\theta$ and $\mu=\rho\sin\theta$.

Again corresponding to Eq. (5), we have

$$\left.\begin{aligned}\overline{A_{nm}\cdot A_{n'm'}}=0\,, \quad n\neq n',\ m\neq m'\pm(\pi)\\ \overline{B_{nm}\cdot B_{n'm'}}=0\,, \quad n\neq n',\ m\neq m'\pm(\pi)\end{aligned}\right\} \tag{28}$$

and corresponding to Eq. (6),

$$\overline{A_{nm}B_{n'm'}}=0 \tag{29}$$

for all n, m, n' and m'.

Using the above formulas we can write the spatial autocorrelation function $\phi(\xi, \eta, t)$ for the two dimensional waves in terms of their spectrum density in space as

$$\begin{aligned}\phi(\xi, \eta, t) &= \overline{u(x, y, t)u(x+\xi, y+\eta, t)}\\ &=\sum\sum\frac{\rho_n\Delta\rho_n\Delta\theta_m}{(2\pi)^2}\Big\{|G^A(\rho_n, \theta_m)|^2\cos^2(c\rho_n t)\\ &\qquad+\frac{|G^B(\rho_n, \theta_m)|^2}{c^2\rho_n^2}\sin^2(c\rho_n t)\Big\}\exp(i\rho_n\xi\cos\theta_m+i\rho_n\eta\sin\theta_m)\end{aligned}$$

From the above equation, it follows that the condition for a stationary stochastic wave of two dimensions is written as

$$\left.\begin{aligned}|G^A(\rho, \theta)|^2&=\frac{|G^B(\rho, \theta)|^2}{c^2\rho^2}\\ \text{or}\qquad \omega_n^2\overline{|A_{nm}|^2}&=\overline{|B_{nm}|^2}\end{aligned}\right\}. \tag{30}$$

Introducing this into $\phi(\xi, \eta, t)$, we obtain

$$\phi(\xi, \eta, t)=\phi(\xi, \eta)=\sum\sum\frac{\rho_n\Delta\rho_n\Delta\theta_m}{(2\pi)^2}|G^A(\rho_n, \theta_m)|^2\exp(i\rho_n\xi\cos\theta_m+i\rho_n\eta\sin\theta_m)$$

Replacing the summation by the integration and dropping the suffix A we have

$$\phi(\xi, \eta)=\frac{1}{(2\pi)^2}\iint|G(\rho, \theta)|^2\exp(i\rho\xi\cos\theta+i\rho\eta\sin\theta)\rho d\rho d\theta \tag{31}$$

and accordingly we also have by the Fourier transformation

$$|G(\rho, \theta)|^2=\iint\phi(\xi, \eta)\exp(-i\rho\xi\cos\theta-i\rho\eta\sin\theta)d\xi d\eta \tag{32}$$

On the other hand, the spectrum density in time of the wave is written as

$$\Phi(\omega_n)=\frac{1}{4}\cdot\frac{\overline{(U_c(\omega_n))^2}+\overline{(U_s(\omega_n))^2}}{\Delta\omega_n/2\pi} \tag{33}$$

where $U_c(\omega_n)$ is the Fourier cosine coefficient of $u(x, y, t)$ with respect to t for given x and y, while $U_s(\omega_n)$ is the corresponding sine coefficient. We see in Eq. (25) that

$$\left.\begin{aligned} U_c(\omega_n)&=\sum_m A_{nm}\exp\left(i\frac{\omega_n}{c}x\cos\theta_m+i\frac{\omega_n}{c}y\sin\theta_m\right)\\ U_s(\omega_n)&=\sum_m \frac{B_{nm}}{\omega_n}\exp\left(i\frac{\omega_n}{c}x\cos\theta_m+i\frac{\omega_n}{c}y\sin\theta_m\right)\\ \omega_n&=cp_n\,. \end{aligned}\right\} \tag{34}$$

Using Eqs. (27), (28), (30), and (32), we can write the spectrum density $\Phi(\omega)$ in terms of the spatial autocorrelation function $\phi(\xi, \eta)$ as follows,

$$\begin{aligned}\Phi(\omega_n)&=\frac{\sum_m |A_{nm}|^2+|B_{nm}|^2/\omega_n^2}{4\Delta\omega_n/2\pi}\\ &=\frac{1}{4\pi c}\int_0^{2\pi}\left|G\left(\frac{\omega_n}{c},\ \theta\right)\right|^2\frac{\omega_n}{c}d\theta\\ &=\frac{1}{4\pi c}\int_0^{2\pi}\frac{\omega_n}{c}d\theta\iint\phi(\xi,\ \eta)\exp\left(-i\frac{\omega_n}{c}\xi\cos\theta-i\frac{\omega_n}{c}\eta\sin\theta\right)d\xi d\eta\,.\end{aligned} \tag{35}$$

Replacing (ξ, η) by a circular coordinate (r, ψ) defined as

$$\xi=r\cos\psi$$
$$\eta=r\sin\psi$$

and using the relation

$$\int_0^{2\pi}d\theta\exp\{-ipr\cos(\theta-\psi)\}=2\pi J_0(pr)$$

we have from Eq. (35)

$$\Phi(\omega)=\frac{1}{2c}\iint\phi(r,\ \psi)J_0(pr)\frac{\omega}{c}drd\psi\,. \tag{36}$$

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If we introduce the azimuthal average of the spatial autocorrelation function, i.e.

$$\overline{\phi}(r)=\frac{1}{2\pi}\int\phi(r,\ \psi)d\psi \qquad (37)$$

we see that a one to one correspondence exists between this function $\overline{\psi}(r)$ and the spectrum density $\Phi(\omega)$ as follows

$$\Phi(\omega)=\frac{\pi}{c^2}\omega\int_0^\infty\overline{\phi}(r)J_0\left(\frac{\omega}{c}r\right)rdr \qquad (38)$$

$$\overline{\phi}(r)=\frac{1}{\pi}\int_0^\infty\Phi(\omega)J_0\left(\frac{\omega}{c}r\right)d\omega \qquad (39)$$

This last equation (39) is derived by the use of the Hankel transformation. It is clear that Eq. (39) corresponds to Eq. (15) for one dimensional waves.

5. Dispersive waves of two dimensions

It will be shown in this section that Eq. (39) also holds in the case of dispersive waves without modifications except the substitution of the function $c(\omega)$ of frequency ω for the constant velocity c.

Taking the relation into account,

$$\Delta\omega_n=\left(\frac{d\omega}{d\rho}\right)_n\Delta\rho_n$$

we obtain

$$\Phi(\omega)=\frac{1}{4\pi}\frac{d\rho}{d\omega}\int_0^{2\pi}\left|G\left(\frac{\omega}{c},\theta\right)\right|^2\frac{\omega}{c}d\theta$$

corresponding to Eq. (35) of non-dispersive waves. From this and Eqs. (32) and (37), we have

$$\Phi(\omega)=\frac{\pi\omega}{c(\omega)}\frac{d\rho}{d\omega}\int_0^\infty\overline{\phi}(r)J_0\left(\frac{\omega}{c(\omega)}r\right)rdr.$$

Then the Hankel transformation yields the final result:

$$\overline{\phi}(r)=\int_0^\infty \frac{c(\omega)}{\pi\omega}\,\frac{d\omega}{d\rho}\,\Phi(\omega)J_0\!\left(\frac{\omega}{c(\omega)}r\right)\frac{\omega}{c(\omega)}\,\frac{d\rho}{d\omega}\,d\omega$$

$$=\frac{1}{\pi}\int_0^\infty \Phi(\omega)J_0\!\left(\frac{\omega}{c(\omega)}r\right)d\omega\ . \qquad (40)$$

6. Spatial autocorrelation of filtered waves (2)

As has been mentioned in Section 3, the measurements in our method are carried out in two steps; first, the seismograph vibrations are filtered, and secondly, among filtered vibrations, the spatial autocorrelation coefficient is computed. So far as we are concerned with waves having a single velocity corresponding to a frequency ω, the azimuthally averaged autocorrelation function $\overline{\phi}(r)$ of the wave filtered by a resonator of frequency ω_0 is written from Eq. (40) as

$$\overline{\phi}(r)=\overline{\phi}(r,\ \omega_0)=P(\omega_0)J_0\!\left(\frac{\omega_0}{c(\omega_0)}\,r\right) \qquad (41)$$

where $P(\omega_0)$ is the same as defined by Eq. (19). In consequence, denoting the corresponding autocorrelation coefficient $\overline{\rho}(r,\ \omega_0)$, we have

$$\overline{\rho}(r,\ \omega_0)=J_0\!\left(\frac{\omega_0}{c(\omega_0)}r\right). \qquad (42)$$

This formula clearly indicates that if one measures $\overline{\rho}(r,\ \omega_0)$ for a certain r and for various ω_0's, he can obtain the function $c(\omega_0)$, i.e. the dispersion curve of the wave for the corresponding range of frequency ω_0.

Now let us proceed to the cases in which waves are polarized, and later refer to the cases in which they are composed of partial waves having different velocities and the above procedure cannot be applied.

7. Polarization

As far as two dimensional waves propagating over a horizontal plane are concerned, it is evident that there is no polarization with respect to the vertical component of vibrations. On the other hand, the horizontal component has two typical modes of polarization; namely the vibration is confined either in the direction parallel to that of propagation, or in the direction perpendicular to that. For instance, in the case of seismic waves, P waves, SV waves, and Rayleigh waves belong to the former, while SH waves and Love waves belong to the latter.

In order to deal with those polarized waves, we observe the component vibration u_r parallel to and the other component u_ψ perpendicular to the direction connecting the two seismometers placed, between the vibrations of which the correlation is to be investigated. Denoting spatial autocorrelation functions for these two components as

$$\phi_r(r, \psi) = \overline{u_r(x, y)u_r(x+r\cos\psi,\ y+r\sin\psi)}$$

$$\phi_\psi(r, \psi) = \overline{u_\psi(x, y)u_\psi(x+r\cos\psi,\ y+r\sin\psi)}$$

we have from Eq. (31) for waves of the parallel polarization

$$\left.\begin{aligned}\phi_r(r, \psi) &= \frac{1}{(2\pi)^2}\iint \cos^2(\theta-\psi)|G(\rho, \theta)|^2 \exp\{i\rho r\cos(\theta-\psi)\}\rho d\rho d\theta \\ \phi_\psi(r, \psi) &= \frac{1}{(2\pi)^2}\iint \sin^2(\theta-\psi)|G(\rho, \theta)|^2 \exp\{i\rho r\cos(\theta-\psi)\}\rho d\rho d\theta\end{aligned}\right\} \tag{43}$$

and in the same way we have for waves of the perpendicular polarization

$$\left.\begin{aligned}\phi_r(r, \psi) &= \frac{1}{(2\pi)^2}\iint \sin^2(\theta-\psi)|G(\rho, \theta)|^2 \exp\{i\rho r\cos(\theta-\psi)\}\rho d\rho d\theta \\ \phi_\psi(r, \psi) &= \frac{1}{(2\pi)^2}\iint \cos^2(\theta-\psi)|G(\rho, \theta)|^2 \exp\{i\rho r\cos(\theta-\psi)\}\rho d\rho d\theta\end{aligned}\right\}. \tag{44}$$

The above equations show that in both cases the sum of two component autocorrelation functions

$$\phi_r(r, \psi) + \phi_\psi(r, \psi)$$

is written in the same form as Eq. (31) for non-polarized waves

$$\phi_r(r, \psi) + \phi_\psi(r, \psi) = \frac{1}{(2\pi)^2}\iint |G(\rho, \theta)|^2 \exp\{i\rho r\cos(\theta-\psi)\}\rho d\rho d\theta. \tag{45}$$

Thus if we know the left hand side of the above equation, we can obtain, by the use of the Fourier transformation, $|G(\rho, \theta)|^2$ which indicates the distribution of direction of wave propagation.

On the other hand, the mode of polarization is shown apparently in the azimuthally averaged autocorrelation functions;

$$\overline{\phi_r}(r)=\frac{1}{2\pi}\int\phi_r(r,\ \psi)d\psi$$

$$\overline{\phi_\psi}(r)=\frac{1}{2\pi}\int\phi_\psi(r,\ \psi)d\psi.$$

From Eq. (43) and the following relations

$$\int_0^{2\pi}\cos^2(\psi-\theta)\exp\{i\rho r\cos(\psi-\theta)\}\,d\psi=\pi\{J_0(\rho r)-J_2(\rho r)\}$$

$$\int_0^{2\pi}\sin^2(\psi-\theta)\exp\{i\rho r\cos(\psi-\theta)\}\,d\psi=\pi\{J_0(\rho r)+J_2(\rho r)\}$$

we obtain for the parallel polarization,

$$\left.\begin{aligned}\overline{\phi_r}(r)&=\frac{1}{2}\frac{1}{(2\pi)^2}\iint|G(\rho,\ \theta)|^2\{J_0(\rho r)-J_2(\rho r)\}\rho d\rho d\theta\\ \overline{\phi_\psi}(r)&=\frac{1}{2}\frac{1}{(2\pi)^2}\iint|G(\rho,\ \theta)|^2\{J_0(\rho r)+J_2(\rho r)\}\rho d\rho d\theta\end{aligned}\right\}. \tag{46}$$

In the similar way, we have for the perpendicular polarization,

$$\left.\begin{aligned}\overline{\phi_r}(r)&=\frac{1}{2}\frac{1}{(2\pi)^2}\iint|G(\rho,\ \theta)|^2\{J_0(\rho r)+J_2(\rho r)\}\rho d\rho d\theta\\ \overline{\phi_\psi}(r)&=\frac{1}{2}\frac{1}{(2\pi)^2}\iint|G(\rho,\ \theta)|^2\{J_0(\rho r)-J_2(\rho r)\}\rho d\rho d\theta\end{aligned}\right\}. \tag{47}$$

If the correlation is taken among the vibrations filtered by a resonator of frequency ω_0, we may write

$$\frac{1}{2\pi}\int|G(\rho,\ \theta)|^2\rho d\theta=P(\omega_0)\delta\left(\rho-\frac{\omega_0}{c(\omega_0)}\right). \tag{48}$$

Then inserting this into Eq. (46), we have the corresponding azimuthally averaged autocorrelation functions for the parallel polarization,

$$\left.\begin{aligned}\overline{\phi_r}(r,\ \omega_0)&=\frac{1}{2}P(\omega_0)\left\{J_0\left(\frac{\omega_0}{c(\omega_0)}r\right)-J_2\left(\frac{\omega_0}{c(\omega_0)}r\right)\right\}\\ \overline{\phi_\psi}(r,\ \omega_0)&=\frac{1}{2}P(\omega_0)\left\{J_0\left(\frac{\omega_0}{c(\omega_0)}r\right)+J_2\left(\frac{\omega_0}{c(\omega_0)}r\right)\right\}\end{aligned}\right\}. \tag{49}$$

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Likewise we have for the perpendicular polarization,

$$\left.\begin{aligned}\overline{\phi}_r(r,\ \omega_0)&=\frac{1}{2}P(\omega_0)\Big\{J_0\Big(\frac{\omega_0}{c(\omega_0)}r\Big)+J_2\Big(\frac{\omega_0}{c(\omega_0)}r\Big)\Big\}\\ \overline{\phi}_\theta(r,\ \omega_0)&=\frac{1}{2}P(\omega_0)\Big\{J_0\Big(\frac{\omega_0}{c(\omega_0)}r\Big)-J_2\Big(\frac{\omega_0}{c(\omega_0)}r\Big)\Big\}\end{aligned}\right\}. \tag{50}$$

From the above formulas it is clear that the measurement of $\overline{\phi}_r(r,\ \omega_0)$ and $\overline{\phi}_\theta(r,\ \omega_0)$ will effectively determine the polarization of the wave.

8. Special cases

In this section we shall consider the following two special cases: (a) $|G(\rho,\ \theta)|^2$ is independent of θ, (b) $|G(\rho,\ \theta)|^2$ is zero except for $\theta=\theta_0$ and $\theta=\theta_0+\pi$. For instance, the case of microtremors will be shown in Chapter 3 to be of the former type, while seismic waves due to earthquakes may belong to the latter if observed at a point distant from their origin. We shall call the former wave an "isotropic wave" and the latter a "plane wave".

In the case of the isotropic wave, writing

$$|G(\rho,\ \theta)|^2=|G(\rho)|^2\,, \tag{51}$$

we get from Eq. (31)

$$\phi(r,\ \psi)=\frac{1}{2\pi}\int_0^\infty|G(\rho)|^2J_0(\rho r)\rho d\rho\,. \tag{52}$$

Thus $\phi(r,\ \psi)$ is independent of ψ, and it is clear that in this case we can replace $\overline{\phi}(r)$ by $\phi(r,\ \psi)$ for an arbitrary ψ in the formulas obtained previously, and we need not take the average of $\phi(r,\ \psi)$ with respect to ψ. This also holds for polarized isotropic waves.

On the other hand, in the case of the plane wave, we may write

$$|G(\rho,\ \theta)|^2=|G'(\rho)|^2\delta(\theta-\theta_0) \tag{53}$$

and we have

$$\phi(r,\psi)=\frac{1}{2\pi}\int |G'(\rho)|^2 \cos\{\rho r\cos(\psi-\theta_0)\}\rho d\rho\,. \tag{54}$$

If we observe the wave filtered by a resonator of frequency ω_0, it follows from Eqs. (48) and (53) that

$$\frac{1}{2\pi}\int |G(\rho,\theta)|^2\rho d\theta = |G'(\rho)|^2\rho = P(\omega_0)\delta\left(\rho-\frac{\omega_0}{c(\omega_0)}\right). \tag{55}$$

Then the corresponding autocorrelation function $\phi(r,\psi,\omega_0)$ is written as

$$\phi(r,\psi,\omega_0)=P(\omega_0)\cos\left\{\frac{\omega_0}{c(\omega_0)}r\cos(\psi-\theta_0)\right\} \tag{56}$$

and also the corresponding autocorrelation coefficient as

$$\rho(r,\psi,\omega_0)=\frac{\phi(r,\psi,\omega_0)}{\phi(0,\psi,\omega_0)}=\cos\left\{\frac{\omega_0}{c(\omega_0)}r\cos(\psi-\theta_0)\right\} \tag{57}$$

or

$$\frac{\cos(\psi-\theta_0)}{c(\omega_0)}=\frac{1}{r\omega_0}\left\{(-1)^n\,\mathrm{Cos}^{-1}\rho(r,\psi,\omega_0)+n\pi\right\}. \tag{58}$$

This last formula (58) shows that we can determine the velocity $c(\omega_0)$ and the angle θ_0 of the direction of propagation by measuring $\rho(r,\psi,\omega_0)$ at two different ψ's, provided the value of n is known beforehand.

Likewise, in the case of polarized plane wave, we have, for instance, for the parallel polarization

$$\left.\begin{aligned}\phi_r(r,\psi,\omega_0)&=\cos^2(\theta_0-\psi)P(\omega_0)\cos\left\{\frac{\omega_0}{c(\omega_0)}r\cos(\psi-\theta_0)\right\}\\ \phi_\psi(r,\psi,\omega_0)&=\sin^2(\theta_0-\psi)P(\omega_0)\cos\left\{\frac{\omega_0}{c(\omega_0)}r\cos(\psi-\theta_0)\right\}\end{aligned}\right\}. \tag{59}$$

In this case, although the velocity and the direction of propagation can still be known after the normalization which is needed to obtain the corresponding autocorrelation coefficient, the mode of polarization cannot be. Therefore, the polarization for an ideal plane wave is better determined by the ordinary method by investigating the amplitudes for various azimuthal angles.

Our method will, however, be effectively applied to a wave composed of two independent waves, which differ from each other in the

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9. Velocity distribution

In this section we shall deal with two dimensional waves of multiple velocities. Supposing that our waves are not polarized, and denoting a quantity related to the n'th component wave by attaching the suffix n, we write our wave in the form,

$$u=\sum_n u_n(x, y, t)\,.$$

Assuming the statistical independence among the component waves as in the case of one dimensional waves we get the following relations,

$$\phi(r, \varphi)=\sum_n \phi_n(r, \varphi)$$

$$\overline{\phi}(r)=\sum_n \overline{\phi}_n(r)$$

$$\overline{\phi}(r, \omega_0)=\sum_n \overline{\phi}_n(r, \omega_0)=\sum_n P_n(\omega_0)J_0\left(\frac{\omega_0}{c_n(\omega_0)}r\right)$$

$$\overline{\rho}(r, \omega_0)=\sum_n \frac{P_n(\omega_0)}{P(\omega_0)}J_0\left(\frac{\omega_0}{c_n(\omega_0)}r\right)\,. \qquad (60)$$

This last equation (60) indicates that if the number N of component waves is finite, we can obtain the velocity and the percentage of power for each component by measuring $\overline{\rho}(r, \omega_0)$ for a given ω_0 and for $(2N-1)$ different r's.

It may happen, as in the case of one dimensional waves, that the velocity of component waves is distributed continuously. For such a wave group, we define the velocity distribution function by the relation,

$$p(\omega_0, c_n)\Delta c_n=P_n(\omega_0)\,,$$

and then we can write

$$\overline{\phi}(r, \omega_0)=\int p(\omega_0, c)J_0\left(\frac{\omega_0}{c(\omega_0)}r\right)dc\,. \qquad (61)$$

From this it follows by the use of the Hankel transformation that

$$\frac{p(\omega_0,\ c)}{P(\omega_0)}=\frac{\omega_0}{c^3}\int_0^\infty \bar{\rho}(r,\ \omega_0)J_0\left(\frac{\omega_0}{c(\omega_0)}r\right)rdr\ . \tag{62}$$

Thus we have the formula by which to determine the velocity distribution function from the value of $\bar{\rho}(r,\ \omega_0)$ for $r=0\sim\infty$ in the case of two dimensional waves. Eq. (62) corresponds to Eq. (24) for one dimensional waves.

In the case of the polarized wave of multiple velocities, we see from the results obtained in Section 7, that if we replace, for instance, $\bar{\phi}(r,\ \omega_0)$ by the sum of the component autocorrelation functions,

$$\bar{\phi}_r(r,\ \omega_0)+\bar{\phi}_\psi(r,\ \omega_0)$$

every formula in this section holds unaltered.

10. Discussions and summary

The results obtained in the preceding sections indicate that the study of waves from the viewpoint of spectrum will give us additional informations which have been neglected because of the lack of proper method of analysis for the purpose. We have dealt with one dimensional stochastic waves in detail, and extended the reasoning followed there to two dimensional waves. It will be easy to proceed to the investigation of three dimensional waves, but this does not seem to be practically necessary for our measurements of waves are usually confined on a plane surface.

We shall enumerate here the principal results obtained in the present chapter.

(1) The spatial autocorrelation coefficient $\rho(\xi,\ \omega_0)$ of a one dimensional wave having a single velocity c and being filtered by a resonator of frequency ω_0 is given by the relation,

$$\rho(\xi,\ \omega_0)=\cos\left(\frac{\omega_0}{c}\xi\right). \tag{21}$$

This holds also for a dispersive wave with the substitution of $c(\omega_0)$ for the constant velocity c.

(2) If we are allowed to assume a continuous distribution of velocity in a stochastic one dimensional wave, we can obtain the velocity distribution function $p(\omega_0,\ c)$ in the form,